Remarks on power series.

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The following three sections are independent of each other. Each section begins with the statement of a theorem and ends with its short proof. These theorems throw sidelights on various aspects of the theory of power series to which the author devoted much of his work almost since the time that he had the privilege to study under the guidance of Professor LEOPOLD FEJÉR and to make his first contacts with Professor FREDERICK RIESZ.

1. *Assume that* $f(z)=e^{-cz}f_1(z)$ *where* $c\geqq 0$ *and* $f_1(z)$ *is an entire function of genus* 0 *having positive zeros only. Let* γ *be the first zero of* $f(z)$ *and put*

$$-zf'(z)/f(z)=s_1z+s_2z^2+s_3z^3+\dots \tag{1}$$

$$1/f(z)=t_0+t_1z+t_2z^2+t_3z^3+\dots \tag{2}$$

Then s_n/s_{n+1} *decrease; and* t_n/t_{n+1} *increases monotonically so that*

$$\frac{t_0}{t_1}\leqq\frac{t_1}{t_2}\leqq\frac{t_2}{t_3}\leqq\dots\leqq\gamma\leqq\dots\leqq\frac{s_2}{s_3}\leqq\frac{s_1}{s_2}. \tag{3}$$

The term "entire function of genus 0" denotes a function of the form

$$f_1(z)=Cz^m\left(1-\frac{z}{\gamma_1}\right)\left(1-\frac{z}{\gamma_2}\right)\left(1-\frac{z}{\gamma_3}\right)\dots \tag{4}$$

$C, m, \gamma_1, \gamma_2, \gamma_3, \dots$ are constants, m an integer, $m\geqq 0$, $0<|\gamma_1|\leqq|\gamma_2|\leqq\dots$, $\Sigma|\gamma_n|^{-1}$ convergent; γ_1 is called the first zero of $f_1(z)$. We do not exclude the case in which the sequence $\gamma_1, \gamma_2, \dots$ is finite or even empty so that, in very special cases, $f_1(z)$ can turn out a polynomial or even a constant. If the γ_n are all positive and $c\geqq 0$, $f(z)=e^{-cz}f_1(z)$ represents the most general function that can be a limit of polynomials with only positive zeros.[1])

[1]) G. PÓLYA, Über Annäherung durch Polynome mit lauter reellen Wurzeln, *Rendiconti del Circolo Mat. Palermo,* **36** (1913), pp. 279—295. To the theory of these functions, started by LAGUERRE and developed by I. SCHUR and the author, I. J. SCHOENBERG added recently an important new chapter; see I. J. SCHOENBERG, On totally positive functions, Laplace integrals and entire functions of the Laguerre—Pólya—Schur type, *Proceedings National Academy of Sciences U. S. A.*, **33** (1947), pp. 11—17; On Pólya frequency functions. II: Variation-diminishing integral operators of the convolution type, *these Acta,* **12 B** (1950), pp. 97—106. Prof. SCHOENBERG communicates me a more general theorem which he found some time ago and from which the present result easily follows.

The hypothesis of our theorem requires that $C \neq 0$, $m=0$ and the sequence $\gamma_1, \gamma_2, \ldots$ has at least one term $\gamma_1 = \gamma$. To the conclusion of our theorem we could give a sharper, but heavier, form by listing the cases of equality in (3) which are few and trivial and will be completely cleared up by the proof.

From the hypothesis of our theorem it follows also that

$$\lim_{n\to\infty} \frac{t_n}{t_{n+1}} = \gamma = \lim_{n\to\infty} \frac{s_n}{s_{n+1}}. \tag{5}$$

These relations are not new. They are due essentially to DANIEL BERNOULLI[2]) whose classical method consists precisely in approximating the minimum root of an equation (as $f(z)=0$) by the ratio of successive coefficients (as s_n/s_{n+1} or t_n/t_{n+1}) of an appropriate power series (as (1) or (2)). Our theorem brings into Bernoulli's method a twofold specialization. First, it chooses a narrow, although quite important, class of functions. Second, it chooses the series (1) and (2). At the price of these specializations, it obtains (3) that is, precise lower and upper bounds for γ at each step of the approximation. Notice that (5) in itself yields no definite estimate for the error of approximation.

Example: Let

$$f(z) = J_0(2z^{\frac{1}{2}}) = 1 - z + \frac{z^2}{4} - \frac{z^3}{36} + \frac{z^4}{576} - \ldots$$

Using just the coefficients displayed, we find

$$\frac{t_3}{t_4} = \frac{304}{211} < \gamma < \frac{48}{33} = \frac{s_3}{s_4}$$

and so for the first root of $J_0(z)$

$$2.4006 < 2\gamma^{\frac{1}{2}} < 2.4121.$$

Proof. Since $f_1(z)$ is of genus 0, s_n, defined by (1), is, for $n \geqq 2$, the sum of the n-th powers of the reciprocal zeros of $f(z)$. Therefore. as well known[3])

$$s_{n+1}^2 \leqq s_n s_{n+2}.$$

For $n \geqq 2$ equality can be attained only if γ is the only, possibly multiple, zero of $f(z)$. Equality for $n=1$ requires, furthermore, $c=0$.

We start discussing t_n.

Lemma. *Assume that* $a_n > 0$ *for* $n \geqq 0$, *that*

$$a_n^2 \geqq a_{n-1} a_{n+1} \tag{6}$$

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[2]) See L. EULER, *Introductio in Analysin Infinitorum,* Opera Omnia, ser. 1, vol. 8, p. 339.

[3]) See *e. g.* G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA, *Inequalities* (Cambridge, 1934), p. 28., theorem 18.

for $n \geqq 1$ and that $p > 0$. Put

$$(1-pz)^{-1} \sum_{0}^{\infty} a_n z^n = \sum_{0}^{\infty} A_n z^n.$$

Then, for $n = 1, 2, 3, \ldots$,

$$A_n^2 > A_{n-1} A_{n+1}.$$

Changing $a_n p^{-n}$, $A_n p^{-n}$ and pz into a_n, A_n and z, respectively, we reduce the theorem to the particular case where $p = 1$, which we assume. Then

$$A_n = a_0 + a_1 + \ldots + a_n,$$

$$A_n^2 - A_{n-1} A_{n+1} = A_n^2 - (A_n - a_n)(A_n + a_{n+1}) = A_n a_n - A_n a_{n+1} + a_n a_{n+1} =$$
$$= a_0 a_n + a_1 a_n + \ldots + a_n a_n - a_0 a_{n+1} - a_1 a_{n+1} - \ldots - a_{n-1} a_{n+1} \geqq a_0 a_n > 0,$$

for it follows from the hypothesis (6) that, for $1 \leqq k \leqq n$,

$$\frac{a_0}{a_1} \leqq \frac{a_1}{a_2} \leqq \ldots \leqq \frac{a_{k-1}}{a_k} \leqq \ldots \leqq \frac{a_{n-1}}{a_n} \leqq \frac{a_n}{a_{n+1}},$$

$$a_k a_n \geqq a_{k-1} a_{n+1}.$$

Now, t_n is defined by (2). If $f(z)$ is a polynomial of the first degree, $t_n^2 = t_{n-1} t_{n+1}$. By repeated application of the Lemma, we derive hence that

(7) $$t_n^2 > t_{n-1} t_{n+1}$$

for polynomials of degree $\geqq 2$. By a limiting process, we pass, not to $f(z)$, but to $f(z)/(z-\gamma)$, which is also a limit of polynomials with only positive roots. If $f(z)/(z-\gamma)$ is not a constant, the coefficients in the expansion of its reciprocal in powers of z are all positive and satisfy, if not (7), an inequality which we obtain from (7) by substituting $\geqq$ for $>$. Now, by another application of the Lemma, we obtain (7) unweakened. In short, (7) holds unless $f(z)$ is a polynomial of the first degree.

2. *Let*

(1) $$a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n + \ldots$$

be a convergent power series but not a polynomial. Then there exists an infinite sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$, where $\varepsilon_n = 1$ or -1 for $n \geqq 0$, such that the function represented by

$$\varepsilon_0 a_0 + \varepsilon_1 a_1 z + \varepsilon_2 a_2 z^2 + \ldots + \varepsilon_n a_n z^n + \ldots$$

satisfies no algebraic differential equation.

This theorem is similar to a theorem stated by FATOU which I proved first[4]).

P r o o f. The theorem is a corollary of an important, almost forgotten, theorem of GRONWALL[5]).

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[4]) A. HURWITZ and G. PÓLYA, Zwei Beweise eines von Herrn Fatou vermuteten Satzes, *Acta Math.*, **40** (1916), pp. 179–183.

[5]) H. GRÖNWALL, Sur les fonctions qui ne satisfont à aucune équation différentielle algébrique, *Öfversigt af Kongl. Vetenskaps-Akademiens Förhandlingar*, Stockholm, 1898, p. 387—395.

Since the series (1) is not a polynomial, there exists a sequence of positive integers $l_1, l_2, l_3, \dots$ such that

$$l_{m+1} > m l_m, \quad a_{l_m} \neq 0,$$

for $m = 1, 2, 3, \dots$ We set $\varepsilon_n = 1$ if $n = l_1$ or l_2 or $l_3, \dots$, $\varepsilon_n = -1$ if n is different from all terms of the sequence $l_1, l_2, l_3, \dots$, and

$$f(z) = \sum_{0}^{\infty} a_n z^n, \quad g(z) = \sum_{n=0}^{\infty} \varepsilon_n a_n z^n, \quad h(z) = 2 \sum_{m=1}^{\infty} a_{l_m} z^{l_m}.$$

Then

(2) $$f(z) + g(z) = h(z).$$

Now by GRONWALL's result, $h(z)$ cannot satisfy any algebraic differential equation. Therefore, at least one of the two functions, $f(z)$ and $g(z)$, cannot satisfy any algebraic differential equation, since, in the opposite case, their sum $h(z)$ would also satisfy one[6]).

3. *A power series which satisfies an algebraic equation formally is necessarily convergent* (and so it must satisfy that equation actually).

The term "formal" must be accurately explained. We are given a fixed positive integer q. A formal power series in $z^{1/q}$ (abbreviated in the following as f. p. s.) is defined by an infinite sequence of complex numbers a_μ,

(1) $$\dots, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, \dots$$

which, however, begins with an infinity of zeros. That is, there is a k such that

(2) $$a_\mu = 0 \text{ for } \mu < k.$$

The f. p. s. defined by the sequence (1) subject to (2) may be written in either form

$$\sum a_\mu z^{\mu/q}, \quad \sum_{\mu=k}^{\infty} a_\mu z^{\mu/q}.$$

A f. p. s. may be written as a finite sum if only a finite number of the a_μ differ from 0, or even as the monomial a_0 if $a_\mu = 0$ for $|\mu| > 0$. Observe that the integer k arising in (2) may be different for different f. p. s. Equality, addition, multiplication and differentiation of f. p. s. are so defined as suggested by the particular case in which the series are convergent:

(3) $$\Sigma a_\mu x^{\mu/q} = \Sigma b_\mu x^{\mu/q} \text{ means } a_\mu = b_\mu \text{ for } \mu = 0, \pm 1, \pm 2, \dots$$

(4) $$\Sigma a_\mu x^{\mu/q} + \Sigma b_\mu x^{\mu/q} = \Sigma (a_\mu + b_\mu) x^{\mu/q},$$

(5) $$\Sigma a_\lambda x^{\lambda/q} \, \Sigma b_\mu x^{\mu/q} = \Sigma c_\nu x^{\nu/q}$$

where

(6) $$c_\nu = \sum_{\lambda=-\infty}^{\infty} a_\lambda b_{\nu-\lambda}$$

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[6]) See E. H. MOORE, Concerning transcendentally transcendental functions, *Math. Annalen*, **48** (1897), pp. 49–74 for general statements about the field formed by the functions satisfying an algebraic differential equation.

(7) $$(\Sigma a_\mu x^{\mu/q})' = \Sigma(\mu/q) a_\mu x^{(\mu-q)/q}.$$

Observe that the sum in (6) is actually finite, by virtue of the condition (2) and of the analogous condition for the b_μ. With these definitions, the f. p. s. form a ring in which the rule holds: *If a product is* 0, *at least one of the factors is necessarily* 0. In fact, assume that neither factor on the left hand side of (5) is 0. Then there exist an l and an m such that

$$a_\lambda = 0 \text{ for } \lambda < l,\ a_l \neq 0;\ b_\mu = 0 \text{ for } \mu < m,\ b_m \neq 0.$$

Then, however, according to (4),

$$c_{l+m} = a_l b_m \neq 0$$

and so the product is different from 0. (Observe that 0 denotes the f. p. s. for which all terms of the sequence (1) are 0.)

Let $P(x, y, y_1, y_2, \ldots, y_n)$ be a polynomial in its $n+2$ variables and w a f. p. s. On the basis of our definitions, the meaning of the equation

(8) $$P(z, w, w', w'', \ldots, w^{(n)}) = 0$$

is completely clear. If (8) holds, we say that w satisfies the differential equation (8) formally — only formally — if w diverges, also actually if w converges. For instance, the f. p. s.

$$w = 1 + 1!\, z + 2!\, z^2 + 3!\, z^3 + \ldots$$

satisfies the differential equation of order 1

$$z^2 w' + (z-1) w + 1 = 0$$

formally and only formally, not actually. It is thinkable that a f. p. s. w satisfies an algebraic equation

(9) $$P(z, w) = 0,$$

which can be regarded as a differential equation of order 0, only formally. Our theorem asserts that this thinkable situation cannot actually arise[7].

P r o o f. Let n be the degree of the equation (9). Let $w_1, w_2, \ldots, w_n$ denote the convergent Puiseux expansions of the n roots of (9) in the neighborhood of the origin and w a f. p. s. which satisfies (9). We can find a suitable integer q such that $w_1, w_2, \ldots, w_n$ and w are all f. p. s. in $z^{1/q}$. We write (9) in the form

(10) $$A(z)(w-w_1)(w-w_2)\ldots(w-w_n) = 0$$

where $A(z)$ is a polynomial which does not vanish identically. Yet, if the product (10) equals 0, one of its factors must equal 0. That is, w coincides with the convergent Puiseux expansion of one of the roots. Q. E. D.

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[7]) The theorem is an extreme case of results announced elsewhere (G. PÓLYA, Sur les séries entières satisfaisant à une équation différentielle algébrique, *Comptes Rendus Acad. Sci. Paris*, **201** (1935), pp. **444—445**), the proof of which, however, has not yet been published.