On the mapping of the unit-circle by polynomials.

By E. EGERVÁRY in Budapest.

1. Consider the set $\{\Pi\}$ of rational polynomials $P(z)$, which have the following properties:

1. The degree of $P(z)$ is not higher than n,
2. $P(0)=0$,
3. $\Re P(z) \geqq -1$ for $|z| \leqq 1$.

As an application of his theory of non-negative trigonometrical polynomials L. FEJÉR has proved that[1])

$$-1 \leqq \Re P(e^{it}) \leqq n. \tag{1}$$

Somewhat later O. SZÁSZ proved the complementary inequality[2])

$$-\operatorname{ctg}\frac{\pi}{2(n+1)} \leqq \Im P(e^{it}) \leqq \operatorname{ctg}\frac{\pi}{2(n+1)}. \tag{2}$$

Hence all the maps of the unit-circle generated by the polynomials $P(z)$ of the set $\{\Pi\}$ lie in the rectangle

$$-1 \leqq \Re w \leqq n, \quad -\operatorname{ctg}\frac{\pi}{2(n+1)} \leqq \Im w \leqq \operatorname{ctg}\frac{\pi}{2(n+1)}.$$

In the present article I wish to determine the precise field of variability of the maps of the unit-circle which are generated by the set $\{\Pi\}$. The chief results may be stated in the following theorems:

Theorem I. *The point-theoric sum Σ of the maps of the unit-circle which are generated by the set $\{\Pi\}$ is a convex region which coincides with the convex hull of the map of the unit-circle generated by the polynomial*

$$P^*(z)=\frac{2}{n+1}\{nz+(n-1)z^2+\ldots+1\,.\,z^n\}, \quad P^*(z)\subset\{\Pi\}. \tag{3}$$

Theorem II. *The supporting function of Σ with respect to 0 is given by*

$$p(\theta)=\sin\frac{n\theta}{n+1}\bigg/\sin\frac{\theta}{n+1}, \quad -\pi \leqq \theta \leqq \pi. \tag{4}$$

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1) L. FEJÉR, Über trigonometrische Polynome, *Journal für die reine und angewandte Math.*, **146** (1913), pp. 53–82.

2) O. SZÁSZ, Über harmonische Funktionen und L-Formen, *Math. Zeitschrift*, **1** (1918), pp. 149–162.

For the special values $\theta=0$ and $\theta=\frac{\pi}{2}$ this yields the above-mentioned results of FEJÉR and SZÁSZ.

The excentricity of a bounded region with respect to an interior point C may be measured by the ratio of the maximum and minimum of the supporting function of its convex hull with respect to C. Adopting this definition we deduce from our former results the

Theorem III. *The excentricity of the map of the unit-circle generated by an arbitrary polynomial $Q(z)$ of degree n with respect to $Q(0)$ cannot exceed n, this maximal excentricity being attained only for $a+bP^*(\varepsilon z)$ $(a, b$ are arbitrary complex numbers, $|\varepsilon|=1)$.*

2. The proof of these theorems will be based on the following theorem of L. FEJÉR[1]).

The set $\{\Pi\}$ of the polynomials $P(z)$ of degree n

$$P(z)=c_1 z+c_2 z^2+\ldots+c_n z^n$$

which map the unit-circle on the half-plane $\Re(z)\geqq -1$, admits the representation

$$P(z)=\sum_{k=1}^{n}(\bar{\gamma}_0\gamma_k+\bar{\gamma}_1\gamma_{k+1}+\ldots+\bar{\gamma}_{n-k}\gamma_n)z^k, \tag{5}$$

where $\gamma_0, \gamma_1 \ldots \gamma_n$ are arbitrary complex parameters subjected only to the restriction

$$|\gamma_0|^2+|\gamma_1|^2+\ldots+|\gamma_n|^2=1. \tag{5'}$$

Hence if $a_1, a_2, \ldots, a_n$ denote arbitrary constants, the maximum (minimum) of the linear form

$$\Re\sum_{k=1}^{n} a_k c_k, \quad \sum_{k=1}^{n} c_k z^k \subset \{\Pi\}$$

is equal to the greatest (least) root λ^* of the characteristic equation

$$\begin{vmatrix} -\lambda & a_1 & a_2 & \ldots & a_n \\ \bar{a}_1 & -\lambda & a_1 & \ldots & a_{n-1} \\ \bar{a}_2 & \bar{a}_1 & -\lambda & \ldots & a_{n-2} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \bar{a}_n & \bar{a}_{n-1} & \bar{a}_{n-2} & \ldots & -\lambda \end{vmatrix}=0.$$

The maximum (minimum) of $\Re\sum_{1}^{n} a_k c_k$ will be attained for the polynomial $P^*(z)=\sum_{1}^{n}(\bar{\gamma}_0^*\gamma_k^*+\bar{\gamma}_1^*\gamma_{k+1}^*+\ldots+\bar{\gamma}_{n-k}^*\gamma_n^*)z^k$, where $\gamma_0^*, \gamma_1^*, \ldots, \gamma_n^*$ are determined by the system of equations

$$
(6) \qquad \begin{aligned}
-\lambda^*\gamma_0^* + a_1\gamma_1^* + \dots + a_n\gamma_n^* &= 0 \\
\bar{a}_1\gamma_0^* - \lambda^*\gamma_1^* + \dots + a_{n-1}\gamma_n^* &= 0 \\
\cdots\cdots\cdots\cdots\cdots & \\
\bar{a}_n\gamma_0^* + \bar{a}_{n-1}\gamma_1^* + \dots - \lambda^*\gamma_n^* &= 0 \\
|\gamma_0^*|^2 + |\gamma_1^*|^2 + \dots + |\gamma_n^*|^2 &= 1.
\end{aligned}
$$

3. We prove first the following

Lemma. *The point-theoric sum Σ of all the maps of the unit-circle generated by the polynomials $P(z)$ of the set $\{\Pi\}$ is a convex region.*

Indeed, if w_1 and w_2 belong to the region Σ then there are polynomials $P_1(z)$ and $P_2(z)$ such that

$$
\begin{aligned}
P_1(z) \subset \{\Pi\},\ w_1 = P_1(z_1),\ |z_1| \leqq 1, \\
P_2(z) \subset \{\Pi\},\ w_2 = P_2(z_2),\ |z_2| \leqq 1.
\end{aligned}
$$

Consider now the polynomial $Q(z) = \dfrac{P_1(z_1 z) + \mu P_2(z_2 z)}{1+\mu}$, $\mu > 0$. Obviously $Q(z) \subset \{\Pi\}$ and $Q(1) = \dfrac{P_1(z_1) + \mu P_2(z_2)}{1+\mu} = \dfrac{w_1 + \mu w_2}{1+\mu}$, consequently, if w_1 and w_2 belong to the region Σ, then the segment of straight line joining them is contained in Σ, q. e. d.

4. In order to determine the supporting function $p(\theta)$ of the convex region Σ with respect to 0, I will make use of the following representation of the supporting function

$$
(7) \qquad p(\theta) = \max_{w \subset \Sigma} \Re\{e^{-i\theta} w\}, \qquad -\pi \leqq \theta \leqq \pi.
$$

The set of the points $w \subset \Sigma$ is given by

$$
w = P(z)\,;\ P(z) \subset \{\Pi\},\ |z| \leqq 1,
$$

hence

$$
p(\theta) = \max \Re\{e^{-i\theta} P(z)\},\ P(z) \subset \{\Pi\},\ |z| \leqq 1.
$$

But the harmonic function $\Re\{e^{-i\theta}P(z)\}$ attains its extremal values on the boundary, therefore

$$
p(\theta) = \max \Re\{e^{-i\theta} P(e^{it})\},\ P(z) \subset \{\Pi\},\ -\pi \leqq t \leqq \pi.
$$

Suppose, that the maximum will be attained for a $P^*(z) \subset \{\Pi\}$ and for $z = e^{it_1}$. Then $P^{**}(z) = P^*(ze^{it_1}) \subset \{\Pi\}$ and $P^{**}(1) = P^*(e^{it_1})$, consequently

$$
p(\theta) = \max \Re\{e^{-i\theta} P(1)\},\ P(z) \subset \{\Pi\}.
$$

The set of the polynomials $P(z)$ belonging to $\{\Pi\}$ is given by

$$
P(z) = \sum_1^n (\bar{\gamma}_0\gamma_k + \bar{\gamma}_1\gamma_{k+1} + \dots + \bar{\gamma}_{n-k}\gamma_n) z^k, \quad \sum_0^n |\gamma_h|^2 = 1
$$

hence

$$
p(\theta) = \max \Re\{e^{-i\theta} P(1)\} = \max \Re\left\{e^{i\theta} \sum_1^n (\bar{\gamma}_0\gamma_k + \dots + \bar{\gamma}_{n-k}\gamma_n)\right\};\ \sum_0^n |\gamma_k|^2 = 1.
$$

We conclude herefrom that the supporting function $p(\theta)$ will be given by the greatest root of the characteristic equation

$$\begin{vmatrix} -\lambda & e^{i\theta} & e^{i\theta} & \dots & e^{i\theta} \\ e^{-i\theta} & -\lambda & e^{i\theta} & \dots & e^{i\theta} \\ e^{-i\theta} & e^{-i\theta} & -\lambda & \dots & e^{i\theta} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ e^{-i\theta} & e^{-i\theta} & e^{-i\theta} & \dots & -\lambda \end{vmatrix} = (-1)^{n+1} \frac{e^{-i\theta}(\lambda + e^{i\theta})^{n+1} - e^{i\theta}(\lambda + e^{-i\theta})^{n+1}}{e^{-i\theta} - e^{i\theta}} = 0.^{3)}$$

The roots of this equation are

$$\lambda_k(\theta) = \sin \frac{n\theta - k\pi}{n+1} \Big/ \sin \frac{\theta + k\pi}{n+1} \qquad (k = 0, 1, \dots, n),$$

and it is obvious that

$$\lambda_0(\theta) > \lambda_1(\theta) > \dots > \lambda_n(\theta) \quad \text{for} \quad -\pi \leqq \theta \leqq \pi.$$

Consequently, the supporting function of the region Σ is given by

$$(8) \qquad p(\theta) = \lambda_0(\theta) = \sin \frac{n\theta}{n+1} \Big/ \sin \frac{\theta}{n+1}, \qquad -\pi \leqq \theta \leqq \pi,$$

and an easy calculation shows that the boundary of the convex region Σ consists of the arc $-\frac{2\pi}{n+1} \leqq t \leqq \frac{2\pi}{n+1}$ of the curve

$$(9) \qquad w = \frac{2}{n+1} \frac{ne^{it} - (n+1)e^{2it} + e^{(n+2)it}}{(1-e^{it})^2}$$

and of the segment $-\operatorname{ctg} \frac{\pi}{n+1} \leqq t \leqq \operatorname{ctg} \frac{\pi}{n+1}$ of the straight line

$$(10) \qquad w = -1 + it.$$

By the determination of the extremal polynomials we avoid the direct solution of the equations (6) and proceed as follows.

Consider the arithmetic mean

$$P^*(z) = \frac{2}{n+1} \{nz + (n-1)z^2 + \dots + 1 \,.\, z^n\}$$

of the partial sums of the geometrical series

$$\frac{2z}{1-z} = 0 + 2z + 2z^2 + 2z^3 + \dots$$

This polynomial maps the unit-circle $|z| \leqq 1$ on a simple, starshaped region, whose supporting function (with respect to 0) is given by

$$\max_{|z| \leqq 1} \Re \{e^{-i\theta} P^*(z)\} = \Re \left\{ e^{-i\theta} P^* \left(e^{\frac{2i\theta}{n+1}} \right) \right\} = \sin \frac{n\theta}{n+1} \Big/ \sin \frac{\theta}{n+1} = p(\theta).$$

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[3]) See f. i. PÓLYA—SZEGŐ, *Aufgaben und Lehrsätze aus der Analysis*, II (Berlin, 1925), p. 99.

Consequently, the map generated by $w=P^{*}(z)$ of the arc $z=e^{it}$ $-\frac{2\pi}{n+1}\leqq t\leqq\frac{2\pi}{n+1}$ coincides with the arc (9) of the boundary of the convex region Σ.

We have proved in this way that the region Σ is identical to the convex hull of the curve $w=P^{*}(e^{it})$ on which the unit-circle $z=e^{it}$ is mapped by the polynomial

$$P^{*}(z)=\frac{2}{n+1}\{nz+(n-1)z^{2}+\ldots+1\,.\,z^{n}\}.$$

Those points of the boundary of Σ which belong to the line-segment (10) will be attained for

$$Q^{*}(z)=\frac{P^{*}\left(ze^{\frac{2\pi i}{n+1}}\right)+\mu P^{*}\left(ze^{-\frac{2\pi i}{n+1}}\right)}{1+\mu}\subset\{\Pi\};\quad z=1,\,0<\mu<\infty.$$

5. It is evident that the supporting function of a convex region with respect to a point C cannot be constant unless the region is a circle whose center coincides with C. Hence it seems to be natural to define the excentricity of a region (with respect to an interior point) as the ratio of the maximum and minimum of the supporting function belonging to its convex hull.

Now let $Q(z)$ be an *arbitrary* polynomial of degree n. Replace $Q(z)$ by $P(z)=\Gamma\{Q(\varepsilon z)-Q(0)\}$ where Γ and ε are so chosen that

$$\min_{|\theta|\leqq\pi}\left[\max_{|z|\leqq 1}\Re\{e^{i\theta}P(z)\}\right]=\Re\{P(1)\}=-1$$

consequently

$$\Re\{P(z)\}\geqq-1\quad\text{for}\quad|z|\leqq 1.$$

From our former results we infer that the supporting function $p(\theta)$ belonging to the convex hull of the region $w=\Gamma\{Q(\varepsilon z)-Q(0)\}$, $|z|\leqq 1$ verifies the following conditions: $p(0)=\min p(\theta)=1$ and

$$\max p(\theta)\leqq\max\left\{\sin\frac{n\theta}{n+1}\Big/\sin\frac{\theta}{n+1}\right\}=n=n\ \min p(\theta).$$

But the excentricity of a region is obviously invariant under the homothetic transformation $P(z)=\Gamma\{Q(\varepsilon z)-Q(0)\}$. Thus the inequality

$$\max p(\theta)\leqq n\ \min p(\theta)$$

is generally proved.

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