The extension of the notion "relatively prime".

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1. Introduction. The concept of relatively prime ideals has for its origin E. Noether's fundamental work "Idealtheorie in Ringbereichen"[1]). Since that time another definition has been given by W. KRULL in his classical paper "Idealtheorie in Ringen ohne Endlichkeitsbedingung"[2]). Krull's definition always coincides with Noether's for elements, but not necessarily for arbitrary ideals; however in rings with maximal condition the two definitions are equivalent.

In a previous paper[3]) I have made an extension of the Noetherian notion "relatively prime" to the concept of "relatively primary". In the present note I define this concept on the basis of Krull's definition of "relatively prime" and I shall show then that the results which were proved in my cited paper merely for rings with maximal condition, may be proved for the most general rings in a much more simplified form. The method is based upon the fundamental concept of isolated primary component which will occupy an important position in our present subject. With the aid of the new definition and formulation one may easily define even the kernel of an ideal.

Our primary aim here is to continue to develop this part of ideal theory by presenting and discussing a new concept, called "almost relatively prime", being a specialization of the notion "relatively primary", but still remaining a proper generalization of the notion "relatively prime". An interesting result is theorem 4 which presents in-

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1) *Math. Annalen,* **83** (1921), pp. 24—66.

2) *Math. Annalen,* **101** (1929), pp. 729—744. Most of our fundamental concepts are here defined: $\mathfrak{p}$ is a *minimal prime ideal of* $\mathfrak{a}$, if $\mathfrak{p}$, but no proper prime multiple of $\mathfrak{p}$ divides $\mathfrak{a}$; $\mathfrak{p}^*$ is a *maximal prime ideal of* $\mathfrak{a}$ if $\mathfrak{p}^*$ contains no element prime to $\mathfrak{a}$ but each proper divisor of $\mathfrak{p}^*$ contains at least one. The *isolated primary component* $\mathfrak{y}$ *of* $\mathfrak{a}$ associated with a minimal prime ideal $\mathfrak{p}$ consists of all elements whose product with a properly chosen element not belonging to $\mathfrak{p}$ lies in $\mathfrak{a}$. The *kernel* $\mathfrak{k}$ *of* $\mathfrak{a}$ is the intersection of all isolated primary components of $\mathfrak{a}$. The *radical* $\mathfrak{r}$ consists of all elements of which a power belongs to $\mathfrak{a}$.

3) On relatively primary ideals, *Det Kgl. Norske Videnskabers Selskabs Forhandlinger,* **20** (1947), pp. 25—28. I have given a far-reaching extension in my paper: Further generalization of the notion of relatively prime ideals, *Bull. Calcutta Math. Society,* **39** (1947), pp. 143—146.

formation about the case when, for an ideal $\mathfrak{a}$, the concepts "prime to $\mathfrak{a}$" and "almost prime to $\mathfrak{a}$" are equivalent. In rings where no ideal has an infinite number of minimal prime ideals, one may characterize the quasi-primary ideals[4]) with the help of the new concepts in two different ways, and in addition, in rings with maximal condition one is able to define the quasi-primary ideals as well as the primary ideals by a negative property.

The main interest of these last characterizations lies in the fact that they are relative ones, concerning one ideal relatively to another.

2. The notion "relatively primary". We shall say that b is *relatively primary to* $\mathfrak{a}$,[5]) if $bc \in \mathfrak{a}$ implies $c \in \mathfrak{r}$ where $\mathfrak{r}$ denotes the radical of $\mathfrak{a}$; further, $\mathfrak{b}$ is called primary to $\mathfrak{a}$ if $\mathfrak{b}$ contains at least one element primary to $\mathfrak{a}$.

Theorem 1. *b is primary to $\mathfrak{a}$ if and only if it belongs to no isolated primary component of* $\mathfrak{a}$.

If no isolated primary component of $\mathfrak{a}$ contains b, then $bc \in \mathfrak{a}$ implies that c must belong to all minimal prime ideals associated with $\mathfrak{a}$, that is[6]), $c \in \mathfrak{r}$. Conversely, if b is primary to $\mathfrak{a}$, and b would belong to the isolated primary component $\mathfrak{y}$ associated with the minimal prime ideal $\mathfrak{p}$, then we could find an element c not in $\mathfrak{p}$ such that $bc \in \mathfrak{a}$. Hence we should get $c \in \mathfrak{r}$, a contradiction to $c \bar{\in} \mathfrak{p}$.

3. A new definition of the kernel. Theorem 1 asserts that if two ideals have the same isolated primary components, then the same elements are primary to them. As KRULL has proved[7]), the isolated primary components of the kernel of $\mathfrak{a}$ coincide with those of $\mathfrak{a}$, therefore, the same elements are primary to an ideal $\mathfrak{a}$ and to its kernel $\mathfrak{k}$. The kernel of $\mathfrak{a}$ is clearly the maximal ideal with this property, hence the kernel may be defined as follows:

Theorem 2. *The kernel of $\mathfrak{a}$ is the maximal ideal to which the same elements are primary as to* $\mathfrak{a}$.

4. The notion "almost relatively prime". We say that b is *almost relatively prime to* $\mathfrak{a}$ if b is prime to the radical $\mathfrak{r}$ of $\mathfrak{a}$, that is, if $bc \in \mathfrak{r}$ implies $c \in \mathfrak{r}$. We call the ideal $\mathfrak{b}$ almost prime to $\mathfrak{a}$ if it contains at least one element almost prime to $\mathfrak{a}$.

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[4]) The quasi-primary ideals are defined in my paper "On quasi-primary ideals", *these Acta,* 11 (1947), pp. 174—183. An ideal $\mathfrak{q}$ is quasi-primary if $ab \in \mathfrak{q}$ implies that some power of a or of b belongs to $\mathfrak{q}$. An equivalent definition is that its radical is a prime ideal.

[5]) For the sake of brevity, when there is no risk of ambiguity, the term "relatively" will be neglected.

[6]) The radical is the intersection of all minimal prime ideals of $\mathfrak{a}$; cf. Krull's cited paper [2]).

[7]) Loc. cit. [2]), Satz 8.

If b is prime to $\mathfrak{a}$, then so is b^n too, consequently, $bc \in \mathfrak{r}$ or $b^n c^n \in \mathfrak{a}$ implies $c^n \in \mathfrak{a}$, $c \in \mathfrak{r}$. Thus the notion "almost prime to $\mathfrak{a}$" may be regarded as an extension of the notion "prime to $\mathfrak{a}$". The extension is in general a proper one, for in the polynomial domain of x and y with rational coefficients, $b = x^2 + xy$ is almost prime to the quasi-primary ideal $\mathfrak{q} = (x^2 y, y^2)$ with the radical (y), but b is not prime to $\mathfrak{q}$, namely, $by \in \mathfrak{q}$ without $y \in \mathfrak{q}$.

5. The connection between the two notions. It is of some interest to exhibit the connection between the two concepts "primary to $\mathfrak{a}$" and "almost prime to $\mathfrak{a}$".

Theorem 3. *b is almost prime to $\mathfrak{a}$ if and only if each power of b is primary to $\mathfrak{a}$.*

If all powers b^n are primary to $\mathfrak{a}$, then $bc \in \mathfrak{r}$, or, what is the same, $b^s c^s \in \mathfrak{a}$ implies that $c^s \in \mathfrak{r}$, $c \in \mathfrak{r}$ in accordance with the hypothesis. On the other hand, if b is almost prime to $\mathfrak{a}$, and if $b^r c \in \mathfrak{a}$, then $bc \in \mathfrak{r}$ and hence, by hypothesis, we may conclude that $c \in \mathfrak{r}$, q. e. d.

We now prove an interesting fact: $\mathfrak{b}$ is almost prime to $\mathfrak{a}$ if and only if the radical $\mathfrak{s}$ of $\mathfrak{b}$ is prime to the radical $\mathfrak{r}$ of $\mathfrak{a}$. Indeed, if $\mathfrak{b}$ contains an element prime to $\mathfrak{r}$, then the same holds for $\mathfrak{s}$ a fortiori, and if $b \in \mathfrak{s}$ is prime to $\mathfrak{r}$, then so is $b^n \in \mathfrak{b}$ too.

6. Ideals for which "prime to" and "almost prime to" are equivalent. From theorems 1 and 3 it is evident that *b is almost prime to $\mathfrak{a}$ if and only if it belongs to no minimal prime ideal of $\mathfrak{a}$.* Hence it is clear that b is prime to or only almost prime to $\mathfrak{a}$ according as b belongs to no maximal prime ideal associated with $\mathfrak{a}$ or only to no minimal one.

If we were merely considering rings in which every prime ideal is divisorless, i. e., has no proper divisor other than the unit ideal, the maximal and minimal prime ideals associated with $\mathfrak{a}$ would coincide, consequently, there would be no difference between the concepts "prime to $\mathfrak{a}$" and "almost prime to $\mathfrak{a}$".

But even in most general rings there are ideals for which these two concepts coincide:

Theorem 4. *All elements almost prime to $\mathfrak{a}$ are prime to $\mathfrak{a}$ if $\mathfrak{a}$ is identical to its kernel.*

If $\mathfrak{k}$ denotes the kernel of $\mathfrak{a}$, then $\mathfrak{a} = \mathfrak{k}$ implies that each element contained in no minimal prime ideal must be prime to all isolated primary components and so necessarily to $\mathfrak{a}$.

In particular, when $\mathfrak{a}$ is a quasi-primary ideal, we get from theorem 4 a necessary condition that a quasi-primary ideal $\mathfrak{q}$ be primary, viz. that each element almost prime to it be prime to it.

7. Two theorems on quasi-primary ideals. In this section let us confine our discussions to rings in which every ideal possesses only a finite number of minimal prime ideals and so only a finite number of isolated primary components. In such rings we may characterize the quasi-primary ideals by the following two theorems[8].

Theorem 5. *$\mathfrak{q}$ is a quasi-primary ideal if and only if the elements not primary to it form an ideal. This ideal is then the unique primary component $\mathfrak{y}$ of $\mathfrak{q}$.*

On account of theorem 1, we have only to prove that if $\mathfrak{q}$ has more than one isolated primary component, $\mathfrak{y}_1, \ldots, \mathfrak{y}_k$ $(k>1)$, then the elements which are not primary to $\mathfrak{q}$ cannot form an ideal. Let a_j $(j=1,\ldots,k)$ be such an element of $\mathfrak{a}_j = \mathfrak{y}_1 \cap \ldots \cap \mathfrak{y}_{j-1} \cap \mathfrak{y}_{j+1} \cap \ldots \cap \mathfrak{y}_k$ which does not belong to $\mathfrak{y}_j$. Such an a_j necessarily exists, for $\mathfrak{p}_j$ associated with $\mathfrak{y}_j$ divides $\mathfrak{y}_j$ but not $\mathfrak{a}_j$. Now $a = a_1 + \ldots + a_k$ is primary to $\mathfrak{q}$, since each term a_m except a_j belongs to $\mathfrak{y}_j$, consequently, a belongs to no isolated primary component of $\mathfrak{q}$. Hence it follows that $\mathfrak{q}$ is either quasi-primary or fails to possess the stated property.

The other theorem on quasi-primary ideals reads as follows.

Theorem 6. *$\mathfrak{q}$ is quasi-primary if and only if the elements not almost prime to $\mathfrak{q}$ form an ideal, namely, its prime radical.*

If $\mathfrak{q}$ with the stated property had more than one minimal prime ideal, $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ $(k>1)$, then we could choose a_j in $\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_{j-1} \cap$ $\cap\, \mathfrak{p}_{j+1} \cap \ldots \cap \mathfrak{p}_k$ but not in $\mathfrak{p}_j$. Now $a = a_1 + \ldots + a_k$ must be almost prime to $\mathfrak{q}$, for a belongs to no minimal prime ideal $\mathfrak{p}_j$.

8. A negative characterization of quasi-primary ideals. Now we impose a further restriction on the ring: henceforth we shall limit our discussions to rings with maximal condition.

An ideal that cannot be represented as the intersection of certain of its proper divisors almost prime to each other is called *almost-prime-indecomposable*. This definition enables us to formulate a condition for quasi-primary ideals, one which yields a negative characterization of quasi-primary ideals.

Theorem 7. *The necessary and sufficient condition that an ideal be almost-prime-indecomposable is that it be quasi-primary*[9]*.*

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[8]) It is an open question whether theorems 5 and 6 are valid in rings without any condition or not.

[9]) That a quasi-primary ideal has always the stated property is a fact which is true in general and is seen from the first part of the proof. We can however assert nothing about the converse when the ring does not satisfy the maximal condition. But, at any rate, the almost-prime-indecomposable ideals may be regarded as a common generalization of quasi-primary and of irreducible ideals.

If $\mathfrak{q} = \mathfrak{c}_1 \cap \ldots \cap \mathfrak{c}_n$ is a quasi-primary ideal under $\mathfrak{p}$ as prime radical, then at least one of $\mathfrak{c}_i$, say $\mathfrak{c}_1$, must have $\mathfrak{p}$ for its radical[10]. The radical $\mathfrak{r}_2$ of $\mathfrak{c}_2$ divides $\mathfrak{p}$ and so it divides $\mathfrak{c}_1$, consequently, $\mathfrak{c}_1$ is not prime to $\mathfrak{r}_2$, $\mathfrak{c}_1$ is not almost prime to $\mathfrak{a}_2$.

On the other hand, if $\mathfrak{q}$ is not quasi-primary, then it may be represented as the shortest intersection of a finite set of quasi-primary ideals, $\mathfrak{q} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_k$ $(k > 1)$ with the prime radicals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ respectively. Since no quasi-primary ideal is here divided by a prime ideal $\mathfrak{p}_i$ with the trivial exception of its own radical, the quasi-primary ideals $\mathfrak{q}_i$ are almost prime to each other. This completes the proof.

9. A negative characterization of primary ideals. We now deal with the problem as to which ideals possess the property to have no representation where at least one of two irredundant components[11] is almost prime to the other. These ideals will be called *semi-almost-prime-indecomposable* ideals. We now proceed to prove

Theorem 8. *An ideal is semi-almost-prime-indecomposable if and only if it is primary*[12].

First we prove the necessity. If $\mathfrak{a}$ is not primary, then in a shortest primary decomposition of $\mathfrak{a}$, $\mathfrak{a} = \mathfrak{y}_1 \cap \ldots \cap \mathfrak{y}_k$, the associated prime ideals are different, and therefore at least one of two radicals is prime to the other.

To prove the sufficiency, it is plainly enough to show that if $\mathfrak{y}$ is primary with $\mathfrak{p}$ as associated prime ideal, then in $\mathfrak{y} = \mathfrak{c}_1 \cap \ldots \cap \mathfrak{c}_k$ each component has either the radical $\mathfrak{p}$ or may be simply omitted. Indeed, replacing each $\mathfrak{c}_i$ by one of its shortest primary representations, we have presented $\mathfrak{y}$ as the intersection of a finite number of primary ideals, and we know that here the primary components associated with a prime ideal different from $\mathfrak{p}$ must be redundant[13].

10. A remark. The method used to prove the last theorem may successfully be applied to the investigation of those ideals which cannot be resolved into components, any two of which have the property that at least one of them is prime to the other. In this case not only the proof but also the enuntiation remains the same, notwithstanding that "almost prime to $\mathfrak{a}$" is a more general notion than "prime to $\mathfrak{a}$".

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[10]) If $\mathfrak{r}_1 \cap \cdots \cap \mathfrak{r}_k = \mathfrak{p}$ is prime, then $\mathfrak{r}_1 \ldots \mathfrak{r}_k \subset \mathfrak{p}$ implies that $\mathfrak{p}$ divides and so equals one of $\mathfrak{r}_i$.

[11]) The irredundance is a requirement which is not omissible, for in the contrary, the prime ideal $(x) = (x) \cap (x, y)$ would be semi-almost-prime *decomposable!*

[12]) Again, the sufficiency holds even in the most general rings; cf. footnote [9]).

[13]) The intersection of irredundant primary components associated with different prime ideals is never primary! See e. g. B. L. VAN DER WAERDEN, *Moderne Algebra*, vol. 2 (2nd ed., Berlin, 1940), p. 32.