On some sequences defined by recurrence.*

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I. 1. We start from the well known fact[1]), that the sequence defined by the recurrence formula

$$a_n = \frac{a_{n-2}+a_{n-1}}{2} \qquad (a_1 \text{ and } a_2 \text{ are arbitrary}) \tag{1}$$

can be written in the form

$$a_n = \frac{a_1+2a_2}{3} + \frac{2}{3}(a_1-a_2)\left(-\frac{1}{2}\right)^{n-1}. \tag{2}$$

In fact, let us make an attempt with $a_n = a+bz^{n-1}$. Substituting this in (1), we obtain $2a+2bz^{n-1} = a+bz^{n-3}+a+bz^{n-2}$; $2z^2-z-1=0$. The roots of this equation are 1 and $-\frac{1}{2}$. Thus $a_n = a+b\left(-\frac{1}{2}\right)^{n-1}$, and in particular $a_1 = a+b$ and $a_2 = a-\frac{b}{2}$. Thus $a = \frac{a_1+2a_2}{3}$, $b = \frac{2}{3}(a_1-a_2)$ and this gives the result announced.

(2) implies that a_n converges and $\lim a_n = \frac{a_1+2a_2}{3}$. [2])

I. 2. This can be proved also without using the explicit form (2). Let $a_1 \leqq a_2$, then we have evidently $a_1 \leqq a_3 \leqq \ldots \leqq a_{2j-1} \leqq a_{2j+1} \leqq \ldots$, $a_2 \geqq a_4 \geqq \ldots \geqq a_{2j} \geqq a_{2j+2} \geqq \ldots$, and $a_{2k-1} \leqq a_{2l}$. So both a_{2j-1} and a_{2j} are convergent, $a_{2j-1} \to \alpha$, $a_{2j} \to A$; $\alpha \leqq A$. But $\alpha < A$ is impossible, for $a_{2j+1} = \frac{1}{2}(a_{2j-1}+a_{2j})$ woud imply $\alpha = \frac{1}{2}(\alpha+A) > \alpha$. Thus $\alpha = A = a = \lim a_n$.

I. 3. The value of the limit a as a function $\mu(a_1, a_2)$ of the initial values a_1, a_2 can be found as follows. It has to satisfy the functional equation $\mu(a_1, a_2) = \mu(a_2, a_3)$, i. e. $\mu(a_1, a_2) = \mu\left(a_2, \frac{1}{2}(a_1+a_2)\right)$.

We might seek μ in the form

$$\mu(x, y) = q_1 x + q_2 y \qquad (q_1+q_2 = 1), \tag{3}$$

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* The essentially new parts of this paper are II. 2 and III. — The parts I and (partly) II. 1 contain wellknown-results which can be found in almost any book on Finite Defferences. They serve here for better understanding of what follows.

[1]) *Cf. e. g.* E. CESÀRO—G. KOWALEWSKI, *Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung* (Leipzig, 1904), p. 105.

[2]) (2) shows that the difference in the approximation a_n of $\lim a_n$ forms a geometric sequence with the quotient $q = -\frac{1}{2}$.

for the linearity of the process implies that of μ and evidently $\mu(a, a)=a$. This gives $q_1 a_1+q_2 a_2=q_1 a_2+q_2 \frac{a_1+a_2}{2}$, thus $q_1=1/3$, $q_2=2/3$, $\mu(a_1, a_2)=\frac{1}{3}(a_1+2a_2)$.

A more elementary proof is the following:

We multiply both sides of (1) by 2 and add a_{n-1} to both sides. We get $a_{n-1}+2a_n=a_{n-2}+2a_{n-1}$. Repeating the recurrence we have finally $a_{n-1}+2a_n=a_1+2a_2$. This gives for $n\to\infty$: $a+2a=a_1+2a_2$, thus $a=\frac{1}{3}(a_1+2a_2)$.[3])

I. 4. Our results hold not only for the arithmetic mean, but also for any "quasi-arithmetic" mean $m(x, y)=f^{-1}\left(\frac{f(x)+f(y)}{2}\right)$ (f^{-1} is the inverse function of f), *e. g.* for the geometric mean where $f(t)=\log t$, the harmonic mean where $f(t)=1/t$ and the root-mean-power where $f(t)=t^n$. In fact, if $c_n=f^{-1}\left(\frac{f(c_{n-2})+f(c_{n-1})}{2}\right)$, $f(c_n)=\frac{f(c_{n-2})+f(c_{n-1})}{2}$, hen $f(c_n)=a_n$ satisfies $a_n=\frac{a_{n-2}+a_{n-1}}{2}$. Thus

$$c_n=f^{-1}\left[\frac{f(c_1)+2f(c_2)}{3}+\frac{2}{3}\left(f(c_1)-f(c_2)\right)\left(-\frac{1}{2}\right)^{n-1}\right];$$

c_n converges and $\lim c_n=f^{-1}\left(\frac{f(c_1)+2f(c_2)}{3}\right)$. *E.g.*, for the geometric mean $c_n=\sqrt{c_{n-2}c_{n-1}}$, we have $c_n=\left[c_1c_2^2\left(\frac{c_2}{c_1}\right)^{\left(-\frac{1}{2}\right)^{n-2}}\right]^{\frac{1}{3}}$, $\lim c_n=\sqrt[3]{c_1c_2^2}$.

II. 1. We generalize our problem as follows. Let

$$a_n=\frac{p_1 a_{n-k}+p_2 a_{n-k+1}+\ldots+p_{k-1}a_{n-2}+p_k a_{n-1}}{p_1+p_2+\ldots+p_{k-1}+p_k} \tag{4}$$

($a_1, a_2, \ldots, a_k$ are arbitrary; $p_1, \ldots, p_k \geqq 0$.)

Let us try the methods of I. 1 and conjecture

$$a_n=a+b_1 z_1^{n-1}+b_2 z_2^{n-1}+\ldots+b_{k-1}z_{k-1}^{n-1}. \tag{5}$$

Substituting (5) in (4) we get

$$(p_1+p_2+\ldots+p_{k-1}+p_k)z^k-p_k z^{k-1}-p_{k-1}z^{k-2}-\ldots-p_2 z-p_1=0.$$

As $z=1$ is a root, we can divide by $(z-1)$ and get

$$(p_1+p_2+\ldots+p_{k-1}+p_k)z^{k-1}+(p_1+p_2+\ldots+p_{k-1})z^{k-2}+\ldots+(p_1+p_2)z+p_1=0. \tag{6}$$

The roots of this equation are the numbers $z_1, z_2, \ldots, z_{k-1}$ occurring in (5).

The constants in (5) are solutions of the linear system

$$a_i=a+b_1 z_1^{i-1}+b_2 z_2^{i-1}+\ldots+b_{k-1}z_{k-1}^{i-1} \qquad (i=1, 2, \ldots, k).$$

[3]) It was St. Fenyő who called the author's attention to the problem I. 3.

Multiplying the i-th equation by $p_1+p_2+\ldots+p_i$ and adding all equations we get

$$\begin{aligned}p_1a_1+(p_1+p_2)a_2+\ldots+(p_1+p_2+\ldots+p_k)a_k=\\=a[p_1+(p_1+p_2)+\ldots+(p_1+p_2+\ldots+p_k)]+\\+b_1[p_1+(p_1+p_2)z_1+\ldots+(p_1+p_2+\ldots+p_k)z_1^{k-1}]+\ldots+\\+b_{k-1}[p_1+(p_1+p_2)z_{k-1}+\ldots+(p_1+p_2+\ldots+p_k)z_{k-1}^{k-1}].\end{aligned}$$

As $z_1, z_2,\ldots,z_{k-1}$ are roots of (6), the coefficients of $b_1, b_2,\ldots, b_{k-1}$ are 0; thus

$$a=\frac{p_1a_1+(p_1+p_2)a_2+\ldots+(p_1+p_2+\ldots+p_k)a_k}{p_1+(p_1+p_2)+\ldots+(p_1+p_2+\ldots+p_k)}=$$
$$=\frac{p_1a_1+(p_1+p_2)a_2+\ldots+(p_1+p_2+\ldots+p_k)a_k}{kp_1+(k-1)p_2+\ldots+2p_{k-1}+p_k}.$$

The coefficients of the equation (6) are positive and decreasing and thus by a well known theorem of ENESTRÖM and KAKEYA[4]) all its roots $z_1,z_2,\ldots,z_{k-1}$ are of an absolute value less than 1. This implies that if $n\to\infty$, all members on the right of (5) tend to 0 except a[5]); thus a_n converges and

$$\lim a_n=a=\frac{p_1a_1+(p_1+p_2)a_2+\ldots+(p_1+p_2+\ldots+p_k)a_k}{p_1+(p_1+p_2)+\ldots+(p_1+p_2+\ldots+p_k)}.$$

If *e. g.* $a_n=(a_{n-k}+a_{n-k+1}+\ldots+a_{n-2}+a_{n-1})/k$, then

$$\lim a_n=\frac{a_1+2a_2+\ldots+(k-1)a_{k-1}+ka_k}{1+2+\ldots+(k-1)+k}=\frac{a_1+2a_2+\ldots+(k-1)a_{k-1}+ka_k}{k(k+1)/2}.$$

II. 2. The convergence of a_n can again be proved also directly. We give the proof not only for arithmetic means, not even only for quasi-arithmetic ones, but for any sequence defined by $a_n=m(a_{n-k}, a_{n-k+1},\ldots, a_{n-1})$ where we postulate only that the mean $m(x_1, x_2,\ldots, x_k)$ be a) *reflexive:* $m(x, x,\ldots, x)=x$, b) *strictly increasing,* c) *continuous.* The first two properties imply also d) *internity:*

$$\min(x_1, x_2,\ldots, x_k)\leqq m(x_1, x_2,\ldots, x_k)\leqq \max(x_1, x_2,\ldots, x_k).$$

To prove the convergence of a_n, consider

$$\alpha_n=\min(a_{n-k}, a_{n-k+1},\ldots, a_{n-1}) \text{ and } A_n=\max(a_{n-k}, a_{n-k+1},\ldots, a_{n-1});$$

clearly $\alpha_n\leqq a_n\leqq A_n$. Using d) and the fact, that if we drop one of the numbers the minimum of the remainder can not be smaller, we get

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[4]) G. ENESTRÖM, Härledning af en allmän formel för antalet pensionärer som vid en godtycklig tidpunkt förefinnas inom en sluten pensionskassa, *Öfversigt af Kongl. Svenska Vetenskaps-Akademien Förhandlingar,* **50** (1893), pp. 405–415; Remarque sur un théorème relatif aux racines de l'équation où tous les coefficients sont réels et positifs, *Tohôku Math. Journal,* **18** (1920), pp. 34–36. S. KAKEYA, On the limits of the roots of an algebraic equation with positive coefficients, *Ibidem,* **2** (1912), pp. 140–142.

[5]) We have counted throughout II. 1. as if (6) had only simple roots, but also the presence of multiple roots makes no difficulty as also $n^jz^n\to 0$ with $n\to\infty$ if $|z|<1$.

$$\alpha_n = \min(a_{n-k}, a_{n-k+1}, \ldots, a_{n-1}) =$$
$$= \min[a_{n-k}, a_{n-k+1}, \ldots, a_{n-1}, m(a_{n-k}, a_{n-k+1}, \ldots, a_{n-1})] =$$
$$= \min[a_{n-k}, a_{n-k+1}, \ldots, a_{n-1}, a_n] \leqq \min(a_{n-k+1}, \ldots, a_{n-1}, a_n) = \alpha_{n+1};$$

thus α_n increases. One sees similarly that A_n decreases. Thus α_n and A_n are both convergent: $\alpha_n \to \alpha$, $A_n \to A$; $\alpha \leqq A$. But $\alpha < A$ is impossible, because if $a_j = \alpha_n$ is the smallest among $a_{n-k}, a_{n-k+1}, \ldots, a_{n-1}$ and $a_i = A_j$ is the greatest among $a_{j-k}, a_{j-k+1}, \ldots, a_{j-1}$, then by b)

$$\alpha_n = \min(a_{n-k}, \ldots, a_{n-1}) = a_j = m(a_{j-k}, a_{j-k+1}, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_{j-1}) \geqq$$
$$\geqq m(\alpha_j, \alpha_j, \ldots, \alpha_j, A_j, \alpha_j, \ldots, \alpha_j).$$

If $n \to \infty$, also $j \to \infty$ and by c) we would have $\alpha \geqq m(\alpha, \alpha, \ldots, \alpha, A, \alpha, \ldots, \alpha) > \alpha$. This is impossible and therefore $\alpha = A$, which completes our proof.

The weighted aritmethic mean $m(x_1, \ldots, x_k) = (\sum p_i x_i) / \sum p_i$ satisfies a), b), c) and this assures the convergence of the sequence (4).

The result of II. 2 holds also if in the recurrence formula $a_n = m(a_{n-1}, \ldots, a_{n-k})$ the mean value function m is not the same for every n, supposed that either only a finite number of mean value functions vary, or if in the infinity of m-s there is only a finite number of functions which do not occur infinitely many times.

Also the analogues of I. 3 and of I. 4 can be constructed similarly as those of I. 1 in II. 1. We leave the details to the reader.

III. We point out the interesting fact, that the theorem of ENESTRÖM and KAKEYA [4]) is a consequence of II. 2 (and equivalent to it).

In fact, every equation with positive decreasing coefficients can be written in the form $(p_1 + p_2 + \ldots + p_{k-1} + p_k) z^{k-1} + (p_1 + p_2 + \ldots + p_{k-1}) z^{k-2} + \ldots + (p_1 + p_2) z + p_1 = 0$. It is immediate that $z = 1$ can not satisfy our equation and so the theorem is proved if we show that the sequence $w_n = z^n$ converges. Of course, it is enough to show that the real part and the imaginary part of w_n are both convergent. If we multiply the equation by $z - 1$ we get $(p_1 + p_2 + \ldots + p_{k-1} + p_k) z^k - p_k z^{k-1} - p_{k-1} z^{k-2} - \ldots - p_2 z - p_1 = 0$ or what is the same $z^n = \dfrac{p_1 z^{n-k} + p_2 z^{n-k+1} + \ldots + p_{k-1} z^{n-2} + p_k z^{n-1}}{p_1 + p_2 + \ldots + p_{k-1} + p_k}$, i. e.

$$w_n = \frac{p_1 w_{n-k} + p_2 w_{n-k+1} + \ldots + p_k w_{n-1}}{p_1 + p_2 + \ldots + p_k}.$$

The real and the imaginary parts of w_n satisfy evidently the same recurrence formula, thus, by II. 2 they are convergent. This completes our proof of the theorem of ENESTRÖM and KAKEYA. (II. 2 holds only for real numbers, therefore we could not apply it directly to w_n.)

The well known direct proof [4]) of the theorem of ENESTRÖM and KAKEYA is of course shorter than that one given above in II. 2 and III, but there is perhaps some interest in the fact, that such seemingly distant domains as the theory of mean values and the theory of algebraic equations are so closely connected.

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