On factorisable groups.

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We shall call the group $\mathfrak{G}$ factorisable by the (proper) subgroups $\mathfrak{H}$ and $\mathfrak{K}$, if $\mathfrak{G}=\mathfrak{H}\mathfrak{K}$. There are several papers dealing with the special case $\mathfrak{H}\cap\mathfrak{K}=E$[1]) ($E$ the unit element) or $\mathfrak{H}\cap\mathfrak{K}=\mathfrak{D}$ where $\mathfrak{D}$ is a normal subgroup of $\mathfrak{G}$[2]). The purpose of the present paper is to deduce quite general results on the same problem by omitting the condition of the normality of $\mathfrak{D}$. We shall restrict ourselves to finite groups but many of our results may be extended to infinite groups.

Let $H, H', \ldots$ and $K, K', \ldots$ denote elements of $\mathfrak{H}$ and $\mathfrak{K}$ respectively.

It is known that if the group $\mathfrak{G}$ of finite order is representable as the product of two subgroups, $\mathfrak{G}=\mathfrak{H}\mathfrak{K}$ where $\mathfrak{H}\cap\mathfrak{K}=E$, then each element of $\mathfrak{G}$ can be represented, precisely once, both in the form HK and in the form KH. In the relation $HK=K'H'$ the elements K' and H' are uniquely defined whenever K and H are given. If K is fixed, then H' together with H runs over all elements of $\mathfrak{H}$. We may thus associate with each K the permutation $\begin{pmatrix} H \\ H' \end{pmatrix}$ of the elements of $\mathfrak{H}$.

The case is more difficult, if $\mathfrak{H}\cap\mathfrak{K}=\mathfrak{D}\neq E$. Even in this case the elements of $\mathfrak{G}$ can be represented in the form HK, and also in the form KH, but these representations are no more unique. Moreover if we fix the element K in $HK=K'H'$ then exactly d solutions of H'

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1) G. ZAPPA, Costruzione dei gruppi prodotto di due dati sottogruppi permutabili tra loro, *Atti Secondo Congresso Unione Mat. Italiana Bologna 1940,* pp. 115—125.

J. SZÉP, Über die als Produkt zweier Untergruppen darstellbaren endlichen Gruppen, *Commentarii Math. Helvetici,* **22** (1948), pp. 31—33; On the structure of groups which can be represented as the product of two subgroups, *these Acta,* **12 A** (1950), pp. 57—61.

L. RÉDEI, Die Anwendung des schiefen Produktes in der Gruppentheorie *Journal für die reine und angewandte Math.,* **188** (1951), pp. 201—227.

2) G. CASADIO, Costruzione di gruppi come prodotto di sottogruppi permutabili, *Rendiconti di Mat. e delle sue applicazioni,* (V) **2**, Fasc. III—IV (1941).

belong to each H, d being the order of $\mathfrak{D}$. We shall show that we may associate, also in this case, with each K a set of permutations $\binom{H}{H'}$ and we shall discuss with these sets the structure of the group $\mathfrak{G}=\mathfrak{H}\mathfrak{K}$. From the results of I we get an explicit theorem and corollary in II.

I.

We shall prove the following

Theorem 1. *Let the finite group $\mathfrak{G}$ be factorisable by two proper subgroups $\mathfrak{H}, \mathfrak{K}$:*

$$\mathfrak{G}=\mathfrak{H}\mathfrak{K}. \tag{1}$$

Then to each $K(\in\mathfrak{K})$ we can find a permutation $H\to H'$ of $\mathfrak{H}$ such that

$$HKH'^{-1}\in\mathfrak{K} \tag{2}$$

for each element $H(\in\mathfrak{H})$. Denote by $[K]$ the set of these permutations (belonging to K) and by $[\mathfrak{K}]$ the totality of the permutations in all sets $[K]$. Then $[\mathfrak{K}]$ is a group; $[E]$ is a normal subgroup of $[\mathfrak{K}]$ (E is the unit element of $\mathfrak{G}$) and each $[K]$ is some coset of $[E]$; furthermore the following homomorphism holds:

$$\mathfrak{K}\sim[\mathfrak{K}]/[E]. \tag{3}$$

Remark. It is known, that

$$\mathfrak{H}\mathfrak{K}=\mathfrak{K}\mathfrak{H}, \tag{4}$$

thus the theorem holds also for $\mathfrak{K}, \mathfrak{H}$ instead of $\mathfrak{H}, \mathfrak{K}$.

From (4) we get

$$HK=K'H', \tag{5}$$

where K', H' are not uniquely determined by H and K. If K is fixed and H runs over all elements of $\mathfrak{H}$, then the elements H' are, in general, not all different. We shall show that we can choose elements K' such that the corresponding elements H' shall be different.

Let $\mathfrak{D}=\mathfrak{H}\cap\mathfrak{K}$ and let d denote the order of $\mathfrak{D}$. First we show that the system of the elements H' (each H' taken with its multiplicity) contains from each coset in the right side of $\mathfrak{H}=\mathfrak{D}+\mathfrak{D}\bar{H}+\dots$ exactly d elements. Indeed the system of the H' contains from no coset $\mathfrak{D}\bar{H}$ more elements than d. For, if we had in (5)

$$H_xK=K_xH'_x \qquad (x=1,\dots,d+1), \tag{6}$$

where $H_1,\dots,H_{d+1}$ are different and $H'_x\in\mathfrak{D}\bar{H}$, then from (6) we should get

$$H_xH_y^{-1}=K_xH'_xH'^{-1}_yK_y^{-1} \qquad (x,y=1,\dots,d+1). \tag{7}$$

This is absurd, because the elements $H'_xH'^{-1}_y$ and hence even the elements $H_xH_y^{-1}=K_xH'_xH'^{-1}_yK_y^{-1}$ all belong to $\mathfrak{D}$ for $x=1,\dots,d+1$, in contradiction to the fact that $\mathfrak{D}$ contains d elements.

We write (5) in the form

(8) $$HK = K'D^{-1}DH'$$

with $D \in \mathfrak{D}$; then $K'D^{-1} \in \mathfrak{K}$, $DH' \in \mathfrak{H}$. The above discussions imply that to each $H \in \mathfrak{H}$ we can select a suitable element $D = D(H)$ such that in (8) DH' together with H runs over all elements of $\mathfrak{H}$.

According to (8) we may associate with each $K \in \mathfrak{K}$ a non-empty set of permutations of the elements of $\mathfrak{H}$. The set of permutations belonging to K will be denoted by $[K]$.

Let $\varrho \in [K]$, $\sigma \in [K']$ be two permutations, $K, K' \in \mathfrak{K}$. Then $HK(\varrho H)^{-1}$, $HK'(\sigma H)^{-1} \in \mathfrak{K}$ (ϱH and σH denote the elemente of $\mathfrak{H}$ into which H passes by the permutation ϱ resp. σ), hence $HKK'(\sigma\varrho H)^{-1} \in K$, i. e. $\sigma\varrho \in [KK']$.

This fact will be expressed in following manner:

(9) $$[K][K'] \subseteq [KK'].$$

From [9] it follows that $[\mathfrak{K}]$ (the totality of the permutations in all sets $[K]$) is a group, further $[E]^2 \subseteqq [E]$, i. e. $[E]$ is a subgroup of $[\mathfrak{K}]$. Furthermore (9) implies that $[K][E] \subseteq [K]$, i. e. $[K][E] = [K]$ and of course dually $[E][K] = [K]$. Hence we get that $[E]$ is a normal subgroup of $[\mathfrak{K}]$ and instead of (9) we can write

(10) $$[K][K'] = [KK'].$$

The correspondence $K \to [K]$ is a many-to-one mapping of $\mathfrak{K}$ onto the factor-group $[\mathfrak{K}]/[E]$, and clearly it is (according to (10)) a homomorphism. This completes the proof of theorem 1.

Using Theorem 1 we prove the following

Lemma. *If the group $\mathfrak{K}$ has an element $K(\neq E)$ for which $[K]$ contains the unit permutation (i. e. $[K] = [E]$) then $\mathfrak{G}$ has a normal subgroup $\mathfrak{N}$ $(\neq E)$ for which the isomorphism*

(11) $$\mathfrak{K}/\mathfrak{N} \cong [\mathfrak{K}]/[E]$$

holds.

Proof. The elements K of $\mathfrak{K}$ with $[K] = [E]$, form a subgroup $\mathfrak{N}$ of $\mathfrak{K}$ which is by (3) normal in $\mathfrak{K}$ and (11) holds. Further, since for K $(\in \mathfrak{N})$ we have

$$(K' =)\, HKH^{-1} \in \mathfrak{K} \qquad (H \in \mathfrak{H}),$$

therefore

$$H^{-1}K'H \in \mathfrak{K} \qquad (H \in \mathfrak{H})$$

and hence $K' \in \mathfrak{N}$. Consequently, $H\mathfrak{N}H^{-1} = \mathfrak{N}$, i. e. $\mathfrak{N}$ is normal subgroup in $\mathfrak{G}\ (= \mathfrak{H}\mathfrak{K})$.

II.

From Theorem 1 we obtain the following.

Theorem 2. *If the group $\mathfrak{G}$ is factorisable by two of its proper subgroups $\mathfrak{H}$ and $\mathfrak{K}$ and $\mathfrak{D} = \mathfrak{H} \cap \mathfrak{K}$ contains a normal subgroup ($\neq E$) of $\mathfrak{H}$ (or $\mathfrak{K}$); than $\mathfrak{G}$ is not simple.*

Proof. Let $\mathfrak{D}'(\subset \mathfrak{D})$ be a normal subgroup of $\mathfrak{H}$ and $\mathfrak{D}' \neq E$. If $D(\neq E) \in \mathfrak{D}'$, then $HDH^{-1} \in \mathfrak{D}' \subset \mathfrak{K}$ $(H \in \mathfrak{H})$, therefore according to Theorem 1 $[D]$ contains the unit permutation $H \rightarrow H$, i. e. $[D] = [E]$. This together with Lemma implies the statement.

The following corollary might be of particular interest.

Corollary. *Let $\mathfrak{G} = \mathfrak{H}\mathfrak{K}$, where $\mathfrak{H}$ and $\mathfrak{K}$ are proper subgroups of $\mathfrak{G}$ and $\mathfrak{H}$ is Abelian, further $\mathfrak{D} = \mathfrak{H} \cap \mathfrak{K} \neq E$. Then $\mathfrak{G}$ is not simple.*

In fact, Theorem 2 proves the assertion, since $\mathfrak{D}$ is a normal subgroup of $\mathfrak{H}$.

Remark. It is easy to see that *every finite group, not a cyclic p-group, is factorisable if it has a subgroup of prime index.*

The authors did not find any finite groups which are not factorisable, except the cyclic p-groups.

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