An individual ergodic theorem for non-commutative transformations.

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In a recent correspondence Professor H. E. ROBBINS has raised the following interesting question. Let Ω be a measure space of finite measure, let $p \geqq 1$, and let $L_p(\Omega)$ be the normed linear space consisting of those measurable functions f on Ω for which $|f| = \left(\int_\Omega |f(\omega)|^p \, d\omega\right)^{1/p} < \infty$. Let φ_1, φ_2 be one-to-one measure preserving maps of Ω onto itself. Is it true that for any $f \in L_p(\Omega)$ the double limit

$$\lim_{m, n \to \infty} (m \cdot n)^{-1} \sum_{\nu_1=0}^{m-1} \sum_{\nu_2=0}^{n-1} f(\varphi_1^{\nu_1} \varphi_2^{\nu_2} \omega)$$

exists almost everywhere on Ω? ROBBINS has pointed out that this double sequence converges in the mean of order p. In fact, this follows readily from the mean ergodic theorem of F. RIESZ[1]). As far as the almost everywhere convergence is concerned, it appears that the known methods for proving the individual ergodic theorem fail unless the transformations φ_1, φ_2 commute. It is curious, however, that a proper combination of known ergodic theorems will yield an affirmative answer to ROBBINS' question in case $p > 1$. The question, as far as I know, is unanswered in the case $p = 1$. In this note we shall demonstrate the

Theorem. *Let $\varphi_1, \ldots, \varphi_k$ be one-to-one measure preserving maps of the measure space Ω onto itself and let $p > 1$. Then for every $f \in L_p(\Omega)$ the multiple sequence*

$$(1) \qquad (m_1 \cdots m_k)^{-1} \sum_{\nu_1=0}^{m_1-1} \cdots \sum_{\nu_k=0}^{m_k-1} f(\varphi^{\nu_1} \cdots \varphi^{\nu_k} \omega)$$

is convergent (as $m_1, \ldots, m_k \to \infty$ independently) almost everywhere on Ω, as well as in the mean of order p. Furthermore, this multiple sequence is dominated by a function in $L_p(\Omega)$.

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[1]) F. RIESZ, Some mean ergodic theorems, *Journal London Math. Soc.*, **13** (1938), pp. 274–278.

For a given function $f \in L_p(\Omega)$ we shall write $\bar{f}$ for the function whose value at the place ω is $\bar{f}(\omega) = |f(\omega)|$, and reserve the symbol $|f|$ for the norm of f as a vector in $L_p(\Omega)$. If A is a transformation in $L_p(\Omega)$ we shall write $A(f, \omega)$ for the value of Af at the point $\omega \in \Omega$. We shall be concerned with the following transformations

$$T_i(f, \omega) = f(\varphi_i \omega), \qquad i = 1, \dots, k,$$

$$U(i, m) = m^{-1} \sum_{\nu=0}^{m-1} T_i^{\nu}, \qquad i = 1, \dots, k; \; m = 1, 2, \dots,$$

$$V(m_1, \dots, m_k) = U(k, m_k) \dots U(2, m_2)\, U(1, m_1)$$

With this terminology then, $V(m_1, \dots, m_k)(f, \omega)$ is the multiple sequence (1). Since φ_i is measure preserving we have $|T_i f| = |f|$ and hence

(2) $$|U(i, m) f| \leqq |f|, \quad |V(m_1, \dots, m_k) f| \leqq |f|.$$

By the mean ergodic theorem of F. Riesz we know that there is a projection operator E_i with

(3) $$\lim_m U(i, m) f = E_i f, \quad f \in L_p(\Omega), \quad i = 1, \dots, k.$$

From (2) and (3) it follows immediately that

(4) $$\lim V(m_1, \dots, m_k) f = E_k \dots E_1 f, \quad f \in L_p(\Omega),$$

for indeed suppose this fact has been established for $k-1$ maps $\varphi_2, \dots, \varphi_k$ and note that

$$\begin{aligned} |(U(k, m_k) \dots U(1, m_1) - E_k \dots E_1) f| &\leqq |U(k, m_k) \dots U(2, m_2) \{U(1, m_1) - E_1\} f| + \\ &+ |\{U(k, m_k) \dots U(2, m_2) - E_k \dots E_2\} E_1 f| \leqq \\ &\leqq |\{U(1, m_1) - E_1\} f| + |\{U(k, m_k) \dots U(2, m_2) - E_k \dots E_2\} E_1 f| \end{aligned}$$

approaches zero by our induction hypothesis. In connection with his proof of equation (3) Riesz has shown that E_i projects $L_p(\Omega)$ onto the manifold $\mathfrak{M}_i$ of those f for which $T_i f = f$ and the complementary projection $E_i' = I - E_i$ projects $L_p(\Omega)$ onto the closure of the manifold $(I - T_i) L_p(\Omega)$. Thus if we define $\mathfrak{N}_i$ as the set of functions of the form $(I - T_i) f$ with $\sup_\omega |f(\omega)| < \infty$, we have

(5) $$\mathfrak{M}_i + \mathfrak{N}_i \text{ is dense in } L_p(\Omega),$$

a fact which will be needed later. Now let $g = (I - T_1) f \in \mathfrak{N}_1$ with $|f(\omega)| \leqq k$. Then $U(1, m)(g, \omega) = m^{-1} [f(\omega) - f(\varphi_1^m \omega)]$, so $|V(m_1, \dots, m_k)(g, \omega)| \leqq 2k/m_1$, and so

(6) $$\lim V(m_1, \dots, m_k)(g, \omega) = 0, \quad \omega \in \Omega, \quad g \in \mathfrak{N}_1.$$

For a function $f \in \mathfrak{M}_1$ we have $f(\varphi_1 \omega) = f(\omega)$, for almost all $\omega \in \Omega$ and thus $U(1, m)(f, \omega) = f(\omega)$ for almost all $\omega \in \Omega$ and all $m = 1, 2, \dots$. Since the theorem is known[2]) to be true for $k=1$, we shall apply induction and assume that is has been proved for the case of $k-1$ transformations $\varphi_2, \dots, \varphi_k$ The induction hypothesis yields then for $f \in \mathfrak{M}_1$ the almost everywhere convergence of the multiple sequence

$$V(m_1, \dots, m_k)(f, \omega) = U(k, m_k) \dots U(2, m_2)(f, \omega).$$

This fact combined with (5) and (6) shows that

(7) For every f in a set dense in $L_p(\Omega)$ the sequence $V(m_1, \dots, m_k)(f, \Omega)$ converges almost everywhere on Ω.

Next we define the operator D_i (not linear) by the equation

$$D_i(f, \omega) = \operatorname*{lub}_{1 \leq m} U(i, m)(\bar{f}, \omega), \qquad i = 1, \dots, k.$$

It has been shown by N. WIENER[3]) that $D_i f \in L_p(\Omega)$ if $f \in L_p(\Omega)$. Hence if $g_1 = D_1(f)$, $g_i = D_i(g_{i-1})$, $i = 2, \dots, k$, it follows that $g_k \in L_p(\Omega)$. Since $U(i, m_i)$ is a positive operation, i. e., it takes positive functions into positive functions, we have

$$\begin{aligned} &U(2, m_2)\, U(1, m_1)(\bar{f}, \omega) \leq U(2, m_2)(D_1(f), \omega) \leq \\ &\qquad \leq D_2(g_1, \omega) = g_2(\omega), \quad \omega \in \Omega, m_1, \quad m_2 = 1, 2, \dots, \\ &U(3, m_3)\, U(2, m_2)\, U(1, m_1)(\bar{f}, \omega) \leq g_3(\omega), \quad \omega \in \Omega, \quad m_1, m_2, m_3 = 1, 2, \dots, \end{aligned}$$

etc. This yields

(8) For every $f \in L_p(\Omega)$ we have $|V(m_1, \dots, m_k)(f, \omega)| \leq g_k(\omega)$, and $g_k \in L_p(\Omega)$.

The weaker statement

(9) For every $f \in L_p(\Omega)$ we have $\sup\limits_{1 \leq m_1, \dots, m_k} |V(m_1, \dots, m_k)(f, \omega)| < \infty$,

together with (7) suffices to prove

(10) For every $f \in L_p(\Omega)$ the $\lim V(m_1, \dots m_k)(f, \omega)$ exists a. e. on Ω.

This final implication is proved in Lemma 7 of a paper by N. DUNFORD and D. S. MILLER[4]). In that lemma we simply take Γ_p to be the set of all $\gamma = (m_1, \dots, m_k)$ with $m_i \geq p$, $i = 1, \dots, k$, and define for $\gamma = (m_1, \dots, m_k) \in \Gamma_1$ the

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[2]) The almost everywhere convergence has been proved by H. KHINTCHINE, Zu Birkhoff's Lösung des Ergodenproblems, *Math. Annalen,* 107 (1933), pp. 285—288. The mean convergence has been proved in the paper of F. RIESZ referred to above. The dominated convergence has been proved by N. WIENER, The ergodic theorem, *Duke Math. Journal,* 5 (1939), pp. 1—18.

[3]) Ibid.

[4]) N. DUNFORD and D. S. MILLER, On the ergodic theorem, *Transactions American Math. Soc.,* 60 (1946), pp. 538—549.

operator $T_\gamma \equiv V(m_1, \ldots, m_k)$. Thus statements (4), (8), and (10) prove the theorem.

It should be mentioned that if f belongs to ZYGMUND'S class defined by $\int_\Omega |f(\omega)| \log^+ |f(\omega)| d\omega < \infty$, then $D_1(f) \in L_1(\Omega)$. This has been shown by N. WIENER[5]). Hence for such an f we see by the maximal ergodic theorem of YOSIDA and KAKUTANI[6]) that

$$|U(2, m_2) U(1, m_1)(f, \omega)| \leqq U(2, m_2)(g_1, \omega) \leqq g_2(\omega),$$

and $g_2(\omega) < \infty$ a. e. on Ω. Thus, since the only place where the hypothesis $p > 1$ entered was in the proof of (10), we may say that in case $k = 2$ we have the $\lim V(m_1, m_2)(f, \omega)$ existing a. e. on Ω providing

$$\int_\Omega |f(\omega)| \log^+ |f(\omega)| d\omega < \infty.$$

This fact has also been proved by ZYGMUND (unpublished).

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5) Ibid.

6) K. YOSIDA and S. KAKUTANI, Birkhoff's ergodic theorem and the maximal ergodic theorem, *Proceedings Imperial Acad. Tokyo,* 15 (1939), pp. 165—168.