Farey series and their connection with the prime number problem. II.

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In a previous paper of the same title[1]) we considered the asymptotical behaviour of sums of the type

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu),$$

where the summation is extended over all fractions of the Farey series of order x. By supposing that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n}{}' f\left(\frac{k}{n}\right) = A$$

exists, the relations

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) \sim A\,\Phi(x) \sim \frac{3A}{\pi^2} x^2$$

have been established; we sharpened this result for functions $f(t)$ having a bounded derivative in $0 \leq t \leq 1$, and showed e. g. that, for many special $f(t)$ of this class, the order of the difference

$$R_f(x) = \sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) - A\,\Phi(x)$$

is connected with the validity of RIEMANN's hypothesis concerning the roots of the zeta-function.

Now we shall discuss in detail the case of the simplest functions $f(t)$ which are continuous for $0 < t \leq 1$ (moreover having derivatives of arbitrary order in this interval), but become *infinite* in the left end-point $t=0$; by applying the Euler-Maclaurin summation formula, we obtain in Part 1 the following relations:

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[1]) MIKOLÁS [1]. (Numbers in brackets [ ] refer to the bibliography at the end of the paper.)

$$\text{(I)}\qquad \sum_{\nu=1}^{\Phi(x)} \log \varrho_\nu = -\Phi(x) + \frac{1}{2}\psi(x) + O(xe^{-c_1(\log x)^\gamma}) =$$

$$= -\Phi(x) + \frac{1}{2}x + O(xe^{-c_2(\log x)^\gamma}),$$

where

$$\psi(x) = \sum_{\substack{p^m \leq x \\ p \text{ prime}}} \log p$$

denotes TCHEBYCHEF's function, $\frac{1}{2} < \gamma < \frac{4}{7}$, and $c_1 > 0, c_2 > 0$ are constants depending on the choice of γ only;

$$\text{(IIa)}\qquad \sum_{\nu=1}^{\Phi(x)} \left(\frac{1}{\varrho_\nu}\right)^\gamma = \frac{1}{1-\lambda}\Phi(x) + \frac{\zeta(\lambda)}{(1+\lambda)\zeta(1+\lambda)} x^{1+\lambda} + O(x) =$$

$$= \frac{3}{\pi^2(1-\lambda)} x^2 + \frac{\zeta(\lambda)}{(1+\lambda)\zeta(1+\lambda)} x^{1+\lambda} + O(x \log x), \qquad \text{if } 0 < \lambda < 1;$$

$$\text{(IIb)}\qquad \sum_{\nu=1}^{\Phi(x)} \left(\frac{1}{\varrho_\nu}\right)^\lambda = \begin{cases} \dfrac{\zeta(\lambda)}{(\lambda+1)\zeta(\lambda+1)} x^{\lambda+1} - \dfrac{3}{\pi^2(\lambda-1)} x^2 + O(x^\lambda), & \text{if } 1 < \lambda < 2, \\ \dfrac{\zeta(\lambda)}{(\lambda+1)\zeta(\lambda+1)} x^{\lambda+1} + O(x^\lambda), & \text{if } \lambda \geq 2; \end{cases}$$

finally

$$\text{(III)}\qquad \sum_{\nu=1}^{\Phi(x)} \frac{1}{\varrho_\nu} = \Phi(x) \left\{ \log x + \left(C - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) \right\} + O(x \log^2 x) =$$

$$= \frac{3}{\pi^2} x^2 \left\{ \log x + \left(C - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right) \right\} + O(x \log^2 x),$$

C denoting EULER's constant.

(I) will be of special interest, in view of the corollary

$$\left(\prod_{\nu=1}^{\Phi(x)} \varrho_\nu\right)^{\frac{1}{\Phi(x)}} \sim \frac{1}{e},$$

and because it will be proved:

RIEMANN's hypothesis is true if and only if the relation

$$\text{(IV)}\qquad \sum_{\nu=1}^{\Phi(x)} (\log \varrho_\nu + 1) - \frac{1}{2}\psi(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

holds for all positive values of ε.

In Part 2 we give a theorem of Tauberian type, which may be regarded as a converse of the relations (IIb):

Suppose that $f(t)$ is non-negative, decreasing for $0 < t \leq 1$ and such that

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) \sim Bx^\alpha,$$

B, α denoting constants with $B>0$, $\alpha>2$, respectively. Then we have

$$f\left(\frac{1}{t}\right) \sim B\frac{\alpha\zeta(\alpha)}{\zeta(\alpha-1)}t^{\alpha-1}.$$

Our method is elementary; we do not need e. g. the complex integral formula

$$\text{(V)}\qquad \frac{1}{2\pi i}\lim_{T\to\infty}\int\limits_{(\alpha+\varepsilon)-iT}^{(\alpha+\varepsilon)+iT}\frac{x^s}{s^2}\frac{F(s)}{\zeta(s)}ds = \sum_{n=1}^{[x]}\sum_{\substack{k\leqq n\\(k,n)=1}} f\left(\frac{k}{n}\right)\log\frac{x}{n}\qquad(\varepsilon>0),$$

where $\alpha\geqq 2$ denotes the absolute convergence abscissa of

$$F(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\sum_{k=1}^{n} f\left(\frac{k}{n}\right).$$

Nevertheless, we mention (V) since, under suitable restrictions, its application gives the formula

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu) \sim \operatorname*{Res}_{s=\alpha}\frac{x^s F(s)}{s\zeta(s)},$$

which discloses the deeper ground of the above asymptotical results.

1. The cases $f(t)\equiv\log t$ and $f(t)\equiv\frac{1}{t^\lambda}$ $(\lambda>0)$.

In what follows fat numbers refer to the theorems or formulae of my paper [1].

Next let us apply[2])

L e m m a 1. *If $f(t)$ is continuous, decreasing and non-negative for $0<t\leqq 1$, and if*

$$\lim_{\varepsilon\to+0}\int\limits_{\varepsilon}^{1} f(t)\,dt=\int\limits_{0}^{1} f(t)dt$$

exists, then we have

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu)\sim\Phi(x)\int\limits_{0}^{1} f(t)dt.$$

For $f(t)\equiv\log t$ we can write

$$\int\limits_{\frac{1}{v}}^{1}\log t\cdot dt=-1-\frac{\log v}{v}+\frac{1}{v}\to -1,$$

when $v\to+\infty$; consequently

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[2]) See MIKOLÁS [1], pp. 101—102.

$$(1) \qquad \sum_{\nu=1}^{\Phi(x)} \log \varrho_\nu \sim -\Phi(x),$$

i. e.

$$\left(\prod_{\nu=1}^{\Phi(x)} \varrho_\nu\right)^{\frac{1}{\Phi(x)}} = \exp\left(\frac{1}{\Phi(x)}\sum_{\nu=1}^{\Phi(x)} \log \varrho_\nu\right) = \exp(-1+o(1)) = \frac{1}{e}\cdot e^{o(1)} \to \frac{1}{e},$$

if $x \to +\infty$.

Lemma 2. *The geometrical mean of* F_x[3]) *is asymptotic to* $1/e$.

To improve the relation (1), we use STIRLING's formula in the following elegant form:[4])

$$(2) \qquad \log n! = n \log n - n + \frac{1}{2}\log n + \log\sqrt{2\pi} - \int_n^\infty \frac{P_1(t)}{t}\,dt$$

with

$$(3) \qquad P_1(t) = -\sum_{k=1}^{\infty} \frac{\sin 2k\pi t}{k\pi}.$$

We need still

Lemma 3 *Let* $\Lambda(n) = \begin{cases} \log p, & \textit{if is a prime or one of its powers,} \\ 0 & \textit{otherwise.} \end{cases}$

We have for Tchebychef's function, defined by

$$\psi(x) = \sum_{n=1}^{[x]} \Lambda(n) = \sum_{\substack{p^m \leq x \\ p \text{ prime}}} \log p,$$

the identities[5])

$$(4) \qquad \psi(x) = \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right)\log n = \sum_{n=1}^{[x]} \mu(n) \log\left[\frac{x}{n}\right]!$$

Proof. By the well-known identity[6])

$$(5) \qquad \Lambda(n) = \sum_{d/n} \mu\left(\frac{n}{d}\right)\log d$$

we can write

$$\sum_{n=1}^{[x]} \sum_{d/n} \mu\left(\frac{n}{d}\right)\log d = \sum_{\substack{d,\,\delta \\ d\delta \leq x}} \mu(\delta)\log d = \sum_{d=1}^{[x]} M\left(\frac{x}{d}\right)\log d = \sum_{\delta=1}^{[x]} \mu(\delta)\log\left[\frac{x}{\delta}\right]!$$

According to TCHUDAKOV's results[7]), we have the estimation

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[3]) As usual, F_x denotes the Farey series of order x.

[4]) See e. g. KNOPP [1], p. 547.

[5]) $\mu(n)$ denotes the Möbius function, $M(x) = \sum_{n=1}^{[x]} \mu(n)$.

[6]) See e. g. HARDY—WRIGHT [1], p. 253.

[7]) Cf. TCHUDAKOV [1], Theorem 3 (p. 599). — [2], pp. 421—422.

(6) $$\psi(x)=x+O(xe^{-c_1(\log x)^\gamma}),$$

where γ denotes any number with $\frac{1}{2}<\gamma<\frac{4}{7}$, and $c_1>0$ depends on the choice of γ only.

On the other hand, as is well-known, the relation

(7) $$\psi(x)-x=O(\sqrt{x}\log^2 x)$$

holds if and only if RIEMANN's hypothesis is true[8]).

Now we may formulate

Theorem 1.

$$\sum_{\nu=1}^{\Phi(x)}\log\varrho_\nu=-\Phi(x)+\frac{1}{2}\psi(x)+O(xe^{-c_2(\log x)^\gamma})=$$
$$=-\Phi(x)+\frac{1}{2}x+O(xe^{-c_3(\log x)^\gamma}),$$

where γ has the meaning as in (6), and $c_2>0$, $c_3>0$ depend upon the choice of γ only.

Proof. We take γ as fixed. By using the identity (4) and (2), we can write

$$\sum_{\nu=1}^{\Phi(x)}\log\varrho_\nu=\sum_{n=1}^{[x]}M\left(\frac{x}{n}\right)\sum_{k=1}^{n}\log\frac{k}{n}=\sum_{n=1}^{[x]}M\left(\frac{x}{n}\right)(\log n!-n\log n)=$$
$$=\sum_{n=1}^{[x]}M\left(\frac{x}{n}\right)\left(-n+\frac{1}{2}\log n+\log\sqrt{2\pi}-\int_n^\infty\frac{P_1(t)}{t}dt\right).$$

Partial integration shows that

(8) $$\int_n^\infty\frac{P_1(t)}{t}dt=-\frac{1}{12n}+2\int_n^\infty\frac{P_3(t)}{t^3}dt=O\left(\frac{1}{n}\right)$$

(9) $$P_3(t)=\sum_{k=1}^{\infty}\frac{2\sin 2k\pi t}{(2k\pi)^3},$$

and so, in view of (5), (6), (4), (8), we obtain

(10) $$\sum_{\nu=1}^{\Phi(x)}\log\varrho_\nu=-\Phi(x)+\frac{1}{2}\psi(x)+\log\sqrt{2\pi}+O\sum_{n=1}^{[x]}\frac{1}{n}\left|M\left(\frac{x}{n}\right)\right|.$$

By discussing the function

$$h(u)=\frac{1}{\sqrt{u}}e^{-c_2(\log x-\log u)^\gamma},$$

it is easy to verify that

(11) $$\frac{1}{\sqrt{u}}e^{-c_2\left(\log\frac{x}{u}\right)^\gamma}\leq e^{-c_2(\log x)^\gamma}$$

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[8]) See LANDAU [2], vol. II., pp. 121, 156.

if $1 \leqq u \leqq \frac{x}{\xi}$, $\xi = \exp\left(\frac{c_2\gamma}{2}\right)^{\frac{1}{1-\gamma}}$ (>1); by use of (19) and (11) it follows for the last sum under (10)

$$
\begin{aligned}
\sum_{n=1}^{[x]} \frac{1}{n}\left|M\left(\frac{x}{n}\right)\right| &= O\sum_{n\leqq x} \frac{1}{n}\cdot\frac{x}{n}\, e^{-c_3\left(\log\frac{x}{n}\right)^\gamma} = \\
&= x\left\{O\sum_{n\leqq\frac{x}{\xi}} \frac{1}{n^2} e^{-c_3\left(\log\frac{x}{n}\right)^\gamma} + O\sum_{\frac{x}{\xi}<n\leqq x} \frac{1}{n^2} e^{-c_3\left(\log\frac{x}{n}\right)^\gamma}\right\} = \\
&= x\left\{O\left(e^{-c_3(\log x)^\gamma}\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}\right) + O\sum_{n>\frac{x}{\xi}}\frac{1}{n^2}\right\} = \\
&= x\left\{O\left(e^{-c_3(\log x)^\gamma}\right) + O\left(\frac{1}{x}\right)\right\} = O\left(xe^{-c_3(\log x)^\gamma}\right).
\end{aligned} \tag{12}
$$

(10), (12) and (6) involve indeed

$$\sum_{\nu=1}^{\Phi(x)} \log \varrho_\nu + \Phi(x) = \frac{1}{2}\psi(x) + O(xe^{-c_3(\log x)^\gamma}) = \frac{1}{2}x + O(xe^{-c_3(\log x)^\gamma}).$$

Theorem 2. *The relation*

$$\sum_{\nu=1}^{\Phi(x)} (\log \varrho_\nu + 1) - \frac{1}{2}\psi(x) = O\left(x^{\frac{1}{2}+c_4\frac{\log\log\log x}{\log\log x}}\right), \tag{13}$$

c_4 denoting a positive constant, is equivalent to Riemann's hypothesis; we may write $O(x^{\frac{1}{2}+\varepsilon})$ (with an arbitrary $\varepsilon>0$) instead of the expression on the right-hand side, and x in place of $\psi(x)$.

Proof. If our assertion holds, (7) and (13) are clearly equivalent so that, in fact, $\psi(x)$ may be replaced by x because of

$$O(\sqrt{x}\log^2 x) = O\left(x^{\frac{1}{2}+2\frac{\log\log x}{\log x}}\right) = O\left(x^{\frac{1}{2}+c_4\frac{\log\log\log x}{\log\log x}}\right).$$

1° Next suppose that RIEMANN's hypothesis is true, then we have, according to (20)

$$M(x) = O\left(x^{\frac{1}{2}}\cdot x^{c_4\frac{\log\log\log x}{\log\log x}}\right) = O\left(\sqrt{x}\cdot\exp\frac{c_4\log x\cdot\log\log\log x}{\log\log x}\right).$$

Since the last exponent is monotonically increasing for $x>x_0(>1)$, there are constants $K>0$ and $\xi \geqq x_0$ such that

$$|M(y)| < K\sqrt{y}\cdot x^{c_4\frac{\log\log\log x}{\log\log x}}$$

whenever $1\leqq y\leqq x$, $x>\xi$.

Thus, by (10), we can write

$$\sum_{\nu=1}^{\Phi(x)}(\log \varrho_\nu+1)-\frac{1}{2}\psi(x)=\log\sqrt{2\pi}+O\sum_{n=1}^{[x]}\frac{1}{n}\frac{\sqrt{x}}{\sqrt{n}}x^{c_4\frac{\log\log\log x}{\log\log x}}=$$
$$=O\left(x^{\frac{1}{2}+c_4\frac{\log\log\log x}{\log\log x}}\right).$$

2^0 Assume that (13), i. e. for every $\varepsilon>0$ the relation

(14) $$\sum_{\nu=1}^{\Phi(x)}(\log \varphi_\nu+1)-\frac{1}{2}\psi(x)=O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

is valid.

Since, by Lemma **1** (**38**) and (5),

$$\sum_{\substack{k=n\\(k,n)=1}}\log\frac{k}{n}+\varphi(n)-\frac{1}{2}\Lambda(n)=\sum_{d/n}\mu\left(\frac{n}{d}\right)\left(\sum_{k=1}^{d}\log\frac{k}{d}+d-\frac{1}{2}\log d\right),$$

the application of Lemma **7** gives

(15) $$\sum_{n=1}^{\infty}\frac{1}{n^s}\Bigg(\sum_{\substack{k\leqq n\\(k,n)=1}}\log\frac{k}{n}+\varphi(n)-\frac{1}{2}\Lambda(n)\Bigg)=$$
$$=\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\cdot\sum_{n=1}^{\infty}\frac{1}{n^s}(\log n!-n\log n+n-\frac{1}{2}\log n);$$

this equality holds, however, for $\sigma>1$ because of

$$\log n!-n\log n+n-\frac{1}{2}\log n=O(1)$$

(cf. (2), (8)).

Considering (2), we obtain from (15)

(16) $$\sum_{n=1}^{\infty}\frac{1}{n^s}\Bigg(\sum_{\substack{k\leqq n\\(k,n)=1}}\log\frac{k}{n}+\varphi(n)-\frac{1}{2}\Lambda(n)\Bigg)-\log\sqrt{2\pi}=$$
$$=-\frac{1}{\zeta(s)}\sum_{n=1}^{\infty}\frac{1}{n^s}\int_n^{\infty}\frac{P_1(t)}{t}dt.$$

Our hypothesis (14) implies, by virtue of Lemma **6**, that the series on the left-hand side is regular for $\sigma>\frac{1}{2}$; the series at the right of (16) is, by (8), (convergent and so) regular for $\sigma>0$. Thus, if we show that

(17) $$\sum_{n=1}^{\infty}\frac{1}{n^s}\int_n^{\infty}\frac{P_1(t)}{t}dt\neq 0$$

for $\sigma>\frac{1}{2}$, it follows from (16) that $\zeta(s)$ has no zeros in this half-plane, i. e. that RIEMANN'S hypothesis holds.

To verify (17), we write (cf. (8.)

$$(18)\qquad \sum_{n=1}^{\infty}\frac{1}{n^s}\int_n^{\infty}\frac{P_1(t)}{t}\,dt=-\frac{1}{12}\zeta(s+1)+2\sum_{n=1}^{\infty}\frac{1}{n^{s+1}}\,n\int_n^{\infty}\frac{P_3(t)}{t^3}\,dt=$$

$$=-\zeta(s+1)\left(\frac{1}{12}-2\sum_{n=1}^{\infty}\frac{b_n}{n^{s+1}}\right).$$

Using the condition

$$\sum\frac{b_n}{n^{s+1}}=\sum\frac{\mu(n)}{n^{s+1}}\cdot\sum\frac{1}{n^{s+1}}\,n\int_n^{\infty}\frac{P_3(t)}{t^3}\,dt$$

(the right-hand series are plainly absolutely convergent for $\sigma>0$), we obtain for the coefficients b_n (cf. Lemma **7**)

$$b_n=\sum_{\delta/n}\mu\left(\frac{n}{\delta}\right)\delta\int_{\delta}^{\infty}\frac{P_3(t)}{t^3}\,dt$$

and so (cf. (**2**), (9))

$$B(u)=\sum_{n=1}^{[u]}|b_n|\leqq\sum_{n=1}^{[u]}\sum_{\delta/n}\delta\left|\int_{\delta}^{\infty}\frac{P_3(t)}{t^3}\,dt\right|=\sum_{n=1}^{[u]}\left[\frac{u}{n}\right]n\left|\int_n^{\infty}\frac{P_1(t)}{t^3}\,dt\right|\leqq$$

$$\leqq\sum_{n=1}^{[u]}\left|\int_n^{\infty}\frac{P_3(t)}{t^3}\,dt\right|<u\sum_{k=1}^{\infty}\frac{2}{(2k\pi)^3}\cdot\sum_{n=1}^{[u]}\int_n^{\infty}\frac{dt}{t^3}<u\,\frac{\zeta(3)}{8\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^2}=u\,\frac{\zeta(3)}{48\pi},$$

$$(19)\qquad \left|\sum_{n=1}^{\infty}\frac{b_n}{n^{s+1}}\right|\leqq\sum_{n=1}^{\infty}\frac{|b_n|}{n^{\sigma+1}}=\sum_{n=1}^{\infty}B(n)\left(\frac{1}{n^{\sigma+1}}-\frac{1}{(n+1)^{\sigma+1}}\right)=(\sigma+1)\int_1^{\infty}\frac{B(u)}{u^{\sigma+2}}\,du<$$

$$<(\sigma+1)\frac{\zeta(3)}{48\pi}\int_1^{\infty}\frac{du}{u^{\sigma+1}}=\left(1+\frac{1}{\sigma}\right)\frac{\zeta(3)}{48\pi}\qquad(\sigma>0).$$

Finally, by (18) and (19), we have for $\sigma>\frac{1}{2}$ (cf. (**42**))

$$\left|\sum_{n=1}^{\infty}\frac{1}{n^s}\int_n^{\infty}\frac{P_1(t)}{t}\,dt\right|\geqq|\zeta(s+1)|\left(\frac{1}{12}-2\left|\sum_{n=1}^{\infty}\frac{b_n}{n^{s+1}}\right|\right)>$$

$$>\frac{1}{12}|\zeta(s+1)|\left(1-\left(1+\frac{1}{\sigma}\right)\frac{\zeta(3)}{2\pi}\right)>\frac{|\zeta(s+1)|}{12}\left(1-\frac{3\zeta(3)}{2\pi}\right)=$$

$$=\frac{|\zeta(s+1)|}{12}(1-0.574\ldots)>0,$$

which completes the proof.

As another interesting example, we discuss the case where $f(t) \equiv \frac{1}{t^\lambda}$ $(\lambda > 0)$.

Let us suppose $0 < \lambda < 1$; then, by

$$\lim_{\varepsilon \to +0} \int_\varepsilon^1 \frac{dt}{t^\lambda} = \lim_{\varepsilon \to +0} \frac{1}{1-\lambda}(1-\varepsilon^{1-\lambda}) = \frac{1}{1-\lambda},$$

Lemma 1 gives

$$\sum_{\nu=1}^{\Phi(x)} \frac{1}{\varrho_\nu^\lambda} \sim \frac{\Phi(x)}{1-\lambda}, \tag{20}$$

relation which holds for $\lambda \leqq 0$ too. (Cf. Theorem **2.**)

To find much sharper results, we use the relations

$$1 + \frac{1}{2} + \dots + \frac{1}{n} = \log n + C + \frac{1}{2n} + O\left(\frac{1}{n^2}\right), \tag{21}$$

$$1 + \frac{1}{2^\lambda} + \dots + \frac{1}{n^\lambda} = \zeta(\lambda) - \frac{1}{\lambda-1} \cdot \frac{1}{n^{\lambda-1}} + \frac{1}{2}\frac{1}{n^\lambda} + O\left(\frac{1}{n^{\lambda+1}}\right) \quad (\lambda>0,\ \lambda \neq 1), \tag{22}$$

which can be obtained simply from the Euler—Maclaurin summation formula.

Theorem 3.

$$\sum_{\nu=1}^{\Phi(x)} \left(\frac{1}{\varrho_\nu}\right)^\lambda = \begin{cases} \frac{1}{1-\lambda}\Phi(x) + \frac{\zeta(\lambda)}{(1+\lambda)\zeta(1+\lambda)} x^{1+\lambda} + O(x) = \\ \quad = \frac{3}{\pi^2(1-\lambda)} x^2 + \frac{\zeta(\lambda)}{(1+\lambda)\zeta(1+\lambda)} x^{1+\lambda} + O(x \log x), & \text{if } 0<\lambda<1, \\ \frac{\zeta(\lambda)}{(\lambda+1)\zeta(\lambda+1)} x^{\lambda+1} - \frac{3}{\pi^2(\lambda-1)} x^2 + O(x^\lambda), & \text{if } 1<\lambda<2, \\ \frac{\zeta(\lambda)}{(\lambda+1)\zeta(\lambda+1)} x^{\lambda+1} + O(x^\lambda), & \text{if } \lambda \geqq 2. \end{cases}$$

Proof. Let $\lambda>0, \lambda \neq 1$.

In view of **(14)**, the use of (22) gives (cf. **(5)**, **(6)**)

$$\begin{aligned} \sum_{\nu=1}^{\Phi(x)} \left(\frac{1}{\varrho_\nu}\right)^\lambda &= \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) \sum_{k=1}^{n} \left(\frac{n}{k}\right)^\lambda = \\ &= \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) n^\lambda \left(\zeta(\lambda) - \frac{1}{\lambda-1} \cdot \frac{1}{n^{\lambda-1}} + \frac{1}{2} \cdot \frac{1}{n^\lambda} + O\left(\frac{1}{n^{\lambda+1}}\right)\right) = \\ &= \zeta(\lambda) \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) n^\lambda - \frac{1}{\lambda-1} \Phi(x) + \frac{1}{2} + O \sum_{n=1}^{[x]} \frac{1}{n} \left| M\left(\frac{x}{n}\right) \right| = \\ &= \zeta(\lambda) \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) n^\lambda - \frac{1}{\lambda-1} \Phi(x) + O(x). \end{aligned} \tag{23}$$

Since, according to **(22)**,

$$\sum_{k=1}^{m} k^\lambda = \int_0^m u^\lambda du + \frac{1}{2} m^\lambda + O\int_0^m u^{\lambda-1} du = \frac{m^{\lambda+1}}{\lambda+1} + O(m^\lambda),$$

we have by **(1)**

$$(24) \quad \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) n^\lambda = \sum_{n=1}^{[x]} \mu(n) \sum_{k=1}^{\left[\frac{x}{n}\right]} k^\lambda = \frac{1}{\lambda+1} \sum_{n=1}^{[x]} \mu(n) \left[\frac{x}{n}\right]^{\lambda+1} + O\left(x^\lambda \sum_{n=1}^{[x]} \frac{1}{n^\lambda}\right).$$

By taking $\left[\frac{x}{n}\right] = \frac{x}{n} - \vartheta$ $(0 \leqq \vartheta = \vartheta(x, n) < 1)$, we see that

$$\sum_{n=1}^{[x]} \mu(n) \left[\frac{x}{n}\right]^{\lambda+1} = \sum_{n=1}^{[x]} \mu(n) \left(\frac{x}{n}\right)^{\lambda+1} + O \sum_{n=1}^{[x]} \left(\frac{x}{n}\right)^\lambda =$$

$$= x^{\lambda+1} \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{\lambda+1}} + x^{\lambda+1} O \sum_{n=[x]+1}^{\infty} \frac{1}{n^{\lambda+1}} + O\left(x^\lambda \sum_{n=1}^{[x]} \frac{1}{n^\lambda}\right).$$

Hence, by **(40)** and (cf. **(27)**)

$$\sum_{n=[x]+1}^{\infty} \frac{1}{n^{\lambda+1}} \sim \int_x^\infty \frac{du}{u^{\lambda+1}} = \frac{1}{\lambda x^\lambda},$$

it follows

$$(25) \qquad \sum_{n=1}^{[x]} \mu(n) \left[\frac{x}{n}\right]^{\lambda+1} = \frac{x^{\lambda+1}}{\zeta(\lambda+1)} + O(x) + O\left(x^\lambda \sum_{n=1}^{[x]} \frac{1}{n^\lambda}\right),$$

so that (23), (24), (25) imply

$$(26) \quad \sum_{\nu=1}^{\Phi(x)} \frac{1}{\varrho_\nu^\lambda} = \frac{\zeta(\lambda)}{(\lambda+1)\zeta(\lambda+1)} x^{\lambda+1} - \frac{1}{\lambda-1} \Phi(x) + O(x) + O\left(x^\lambda \sum_{n=1}^{[x]} \frac{1}{n^\lambda}\right).$$

But (cf. **(27)**)

$$\sum_{n=1}^{[x]} \frac{1}{n^\lambda} \sim \begin{cases} \zeta(\lambda) & \text{for } \lambda > 1, \\ \displaystyle\int_1^x \frac{dt}{t^\lambda} \sim \frac{x^{1-\lambda}}{1-\lambda} & \text{for } 0 < \lambda < 1, \end{cases}$$

and therefore we can write

$$\sum_{\nu=1}^{\Phi(x)} \frac{1}{\varrho_\nu^\lambda} - \frac{\zeta(\lambda)}{(\lambda+1)\zeta(\lambda+1)} x^{\lambda+1} - \frac{1}{\lambda-1} \Phi(x) = \begin{cases} O(x^\lambda), & \text{if } \lambda > 1, \\ O(x), & \text{if } 0 < \lambda < 1. \end{cases}$$

Using now **(7)**, we obtain the estimations in question.

Let us take, finally, $\lambda = 1$.

Theorem 4.

$$\sum_{\nu=1}^{\Phi(x)} \frac{1}{\varrho_\nu} = \Phi(x) \left\{\log x + \left(C - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right)\right\} + O(x \log^2 x) =$$

$$= \frac{3}{\pi^2} x^2 \left\{\log x + \left(C - \frac{1}{2} - \frac{\zeta'(2)}{\zeta(2)}\right)\right\} + O(x \log^2 x),$$

where C is Euler's constant.

Proof. Applying (4), (21), we get (cf. (5), (6))

$$\sum_{\nu=1}^{\Phi(x)} \frac{1}{\varrho_\nu} = \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) \sum_{k=1}^{n} \frac{n}{k} = \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) n\left(\log n + C + \frac{1}{2n} + O\left(\frac{1}{n^2}\right)\right) =$$

(27) $$= \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) n \log n + C\Phi(x) + \frac{1}{2} + O \sum_{n=1}^{[x]} \frac{1}{n} \left| M\left(\frac{x}{n}\right)\right| =$$

$$= \sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) n \log n + C\Phi(x) + O(x).$$

Since, according to (27),

$$\sum_{k=1}^{n} k \log k = \sum_{k=1}^{m-1} (k+1) \log (k+1) = \int_0^{m-1} (u+1) \log (u+1)\, du + \frac{1}{2} m \log m +$$

$$+ \frac{B_2}{2!} \{(\log m + 1) - 1\} + \frac{B_4}{4!}\left(1 - \frac{1}{m^2}\right) - 6 \int_0^{m-1} \frac{P_5(u)}{(u+1)^4}\, du =$$

$$= \frac{1}{2} m^2 \log m - \frac{1}{4} m^2 + \frac{1}{2} m \log m + \frac{1}{12} \log m + O(1),$$

we can write, by (4), for the first sum under (27)

$$\sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) n \log n = \sum_{n=1}^{[x]} \mu(n) \sum_{k=1}^{\left[\frac{x}{n}\right]} k \log k =$$

$$= \sum_{n=1}^{[x]} \mu(n) \left\{ \frac{1}{2} \left[\frac{x}{n}\right]^2 \log \left[\frac{x}{n}\right] - \frac{1}{4} \left[\frac{x}{n}\right]^2 + \frac{1}{2} \left[\frac{x}{n}\right] \log \left[\frac{x}{n}\right] + \frac{1}{12} \log \left[\frac{x}{n}\right] + O(1) \right\}.$$

By taking $\left[\frac{x}{n}\right] = \frac{x}{n} - \vartheta$ $(0 \leq \vartheta = \vartheta(x, n) < 1)$, we see that

$$\log \frac{x}{n} - \log \left[\frac{x}{n}\right] = \log\left(1 + \frac{\vartheta}{\left[\frac{x}{n}\right]}\right) < \frac{1}{\left[\frac{x}{n}\right]},$$

$$\log \left[\frac{x}{n}\right] = \log x - \log n - \frac{\theta}{\left[\frac{x}{n}\right]} \quad (0 \leq \theta = \theta(x, n) < 1),$$

and therefore (cf. (5), (6), (4))

$$\sum_{n=1}^{[x]} M\left(\frac{x}{n}\right) n \log n = \sum_{n=1}^{[x]} \mu(n) \left\{ \frac{1}{2} \left[\frac{x}{n}\right]^2 \left(\log x - \log n - \frac{\theta}{\left[\frac{x}{n}\right]}\right) - \frac{1}{4} \left[\frac{x}{n}\right]^2 + \right.$$

(28) $$\left. + \frac{1}{2} \left[\frac{x}{n}\right] \left(\log x - \log n - \frac{\theta}{\left[\frac{x}{n}\right]}\right) + \frac{1}{12} \left(\log x - \log n - \frac{\theta}{\left[\frac{x}{n}\right]}\right) + O(1) \right\} =$$

$$=\log x\left(\Phi(x)-\frac{1}{2}\right)-\frac{1}{2}\sum_{n=1}^{[x]}\mu(n)\log n\left[\frac{x}{n}\right]^2+O\sum_{n=1}^{[x]}\left[\frac{x}{n}\right]-\frac{1}{2}\left(\Phi(x)-\frac{1}{2}\right)+$$
$$+\frac{1}{2}\log x+\frac{1}{2}\psi(x)+O\sum_{n\leqq x}1+\frac{1}{12}M(x)\log x+O\sum_{n\leqq x}\log n+$$
$$+O\sum_{n\leqq x}1+O(x)=\left(\log x-\frac{1}{2}\right)\Phi(x)-\frac{1}{2}\sum_{n=1}^{[x]}\mu(n)\log n\cdot\left[\frac{x}{n}\right]^2+O(x\log x).$$

Considering that (cf. **(40)**)

$$-\frac{d}{ds}\frac{1}{\zeta(s)}=\frac{\zeta'(s)}{\zeta^2(s)}=\sum_{n=1}^{\infty}\frac{\mu(n)\log n}{n^s}\quad(\sigma>1),$$

furthermore by (cf. **(27)**)

$$\sum_{n=[x]+1}^{\infty}\frac{\log n}{n^2}\sim\int_x^{\infty}\frac{\log t}{t^2}dt=\frac{\log x+1}{x},$$

and

$$\sum_{n=1}^{[x]}\frac{\log n}{n}\sim\int_1^{x}\frac{\log t}{t}dt=\frac{1}{2}\log^2 x,$$

we have under (28)

(29) $$\sum_{n=1}^{[x]}\mu(n)\log n\left[\frac{x}{n}\right]^2=\sum_{n=1}^{[x]}\mu(n)\log n\left(\frac{x}{n}-\vartheta\right)^2=$$
$$=x^2\sum_{n=1}^{\infty}\frac{\mu(n)\log n}{n^2}-x^2\,O\sum_{n=[x]+1}^{\infty}\frac{\log n}{n^2}-2x\,O\sum_{n=1}^{[x]}\frac{\log n}{n}+O\sum_{n=1}^{[x]}\log n=$$
$$=x^2\frac{\zeta'(2)}{\zeta^2(2)}+O(x\log x)+O(x\log^2 x)+O(x\log x)=\frac{\zeta'(2)}{\zeta^2(2)}x^2+O(x\log^2 x).$$

Lastly, in view of (27), (28), (29), it follows

$$\sum_{\nu=1}^{q(x)}\frac{1}{\varrho_\nu}=C\Phi(x)+O(x)+\left(\log x-\frac{1}{2}\right)\Phi(x)+O(x\log x)-$$
$$-\frac{\zeta'(2)}{2\zeta^2(2)}x^2+O(x\log^2 x)=\left(\log x+C-\frac{1}{2}\right)\Phi(x)-\frac{\zeta'(2)}{2\zeta^2(2)}x^2+O(x\log^2 x),$$

which by

$$\Phi(x)=\frac{3}{\pi^2}x^2+O(x\log x)=\frac{x^2}{2\zeta(2)}+O(x\log x)$$

(cf. **(7)**), establishes the assertion.

2. A theorem of Tauberian type.

The question arises: how far the asymptotical behaviour of $f(t)$ is determined by that of $\Sigma f(\varrho_\nu)$, i. e. which asymptotical properties of $f(t)$ can be deduced, if an asymptotic formula for $\Sigma f(\varrho_\nu)$ is given?

From this point of view (as a converse of Theorem 3) the following result has a certain interest:

Theorem 5. *Suppose that* $f(t)$ *is non-negative and decreasing for* $0<t\leqq 1$, *furthermore*

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu)\sim Ax^\alpha,$$

α, A *denoting constants with* $\alpha>2, A>0$, *respectively. Then we have*

$$f\left(\frac{1}{t}\right)\sim A\,\frac{\alpha\zeta(\alpha)}{\zeta(\alpha-1)}\,t^{\alpha-1}.$$

For proving this theorem we need the well-known[9])

Lemma 4. *Let* $h(x)$ *be a function which is defined for* $x\geqq 1$ *and let us put*

$$g(x)=\sum_{n=1}^{[x]} h\left(\frac{x}{n}\right). \tag{30}$$

Then we have for $x\geqq 1$,

$$h(x)=\sum_{n=1}^{[x]} \mu(n)\,g\left(\frac{x}{n}\right). \tag{31}$$

Conversely: if (31) *is valid for all* $x\geqq 1$, *then* (30) *holds also.*

Proof of Theorem 5.

1^0 Assume that $f(t)$ satisfies our conditions and

$$\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu)\sim Ax^\alpha \qquad (A>0,\ \alpha>2). \tag{32}$$

Put

$$V(x)=\sum_{k=1}^{[x]} f\left(\frac{k}{[x]}\right), \tag{33}$$

$$G(x)=\sum_{n=1}^{[x]} V(n)=\sum_{n=1}^{[x]}\sum_{k=1}^{n} f\left(\frac{k}{n}\right); \tag{34}$$

then (4) implies

$$H(x)=\sum_{\nu=1}^{\Phi(x)} f(\varrho_\nu)=\sum_{n=1}^{[x]} M\left(\frac{x}{n}\right)V(n)=\sum_{n=1}^{[x]}\mu(x)\,G\left(\frac{x}{n}\right)\sim Ax^\alpha, \tag{35}$$

so that, in virtue of Lemma 4,

$$G(x)=\sum_{n=1}^{[x]} H\left(\frac{x}{n}\right). \tag{36}$$

Let any $\varepsilon>0$ be given. According to (32), a number $\xi=\xi(\varepsilon)\geqq 1$ can be found such that for all $x\geqq\xi$

$$|H(x)-Ax^\alpha|<\varepsilon x^\alpha$$

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[9]) See e.g. LANDAU [1], p. 579.

and therefore for $x \geqq \xi$; $n \leqq \frac{x}{\xi}$,

(37) $$\left| H\left(\frac{x}{n}\right) - A\frac{x^{\alpha}}{n^{\alpha}} \right| < \varepsilon \left(\frac{x}{n}\right)^{\alpha}.$$

Furthermore, by $H(x) = O(x^{\alpha})$, there is a constant $B > 0$ such that

(38) $$\left| H\left(\frac{x}{n}\right) \right| < B\left(\frac{x}{n}\right)^{\alpha} \text{ when } \frac{x}{n} \geqq 1.$$

Now let $x \geqq \xi$ and let us write (cf. (36))

(39) $$G(x) = \sum_{n \leqq \xi} A \cdot \frac{x^{\alpha}}{n^{\alpha}} + \sum_{n \leqq \frac{x}{\xi}} \left(H\left(\frac{x}{n}\right) - A \cdot \frac{x^{\alpha}}{n^{\alpha}} \right) + \sum_{\frac{x}{\xi} < n \leqq x} \left(H\left(\frac{x}{n}\right) - A\frac{x^{\alpha}}{n^{\alpha}} \right) =$$
$$= \Sigma_1 + \Sigma_2 + \Sigma_3.$$

Plainly:

(40) $$\Sigma_1 = A \cdot x^{\alpha} \cdot \sum_{n \leqq x} \frac{1}{n^{\alpha}} = Ax^{\alpha}(\zeta(\alpha) + o(1)) = A\zeta(\alpha) \cdot x^{\alpha} + o(x^{\alpha}).$$

On the other hand, using (37) we get

(41) $$|\Sigma_2| \leqq \sum_{n \leqq \frac{x}{\xi}} \left| H\left(\frac{x}{n}\right) - A \cdot \frac{x^{\alpha}}{n^{\alpha}} \right| < \varepsilon \sum_{n \leqq \frac{x}{\xi}} \left(\frac{x}{n}\right)^{\alpha} < \varepsilon x^{\alpha} \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} = \varepsilon \zeta(\alpha) x^{\alpha}.$$

Finally, according to (38),

(42) $$|\Sigma_3| < \sum_{\frac{x}{\xi} < n \leqq x} \left(\left| H\left(\frac{x}{n}\right) \right| + A \cdot \frac{x^{\alpha}}{n^{\alpha}} \right) \leqq K \sum_{\frac{x}{\xi} < n \leqq x} \left(\frac{x}{n}\right)^{\alpha} < K \sum_{\frac{x}{\xi} < n \leqq x} \xi^{\alpha} \leqq K \xi^{\alpha} x,$$

where $K = \operatorname{Max}(A, B)$.

(39), (40), (41), (42) involve

$$\left| \frac{G(x)}{x^{\alpha}} - A \cdot \zeta(\alpha) \right| < |o(1)| + \varepsilon\zeta(\alpha) + \frac{K\ \xi^{\alpha}}{x^{\alpha+1}}.$$

We see that a number $\eta = \eta(\varepsilon) > \xi$ can be found such that the right-hand sum is less than $c\varepsilon$ (where c is a constant depending only on α), provided that $x \geqq \eta$; this implies

(43) $$G(x) \sim A\zeta(\alpha) \cdot x^{\alpha}.$$

2° We have by hypothesis

(44) $$V(n+1) - V(n) = \sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) - \sum_{k=1}^{n} f\left(\frac{k}{n}\right) =$$
$$= \sum_{n=1}^{n} \left(f\left(\frac{k}{n+1}\right) - f\left(\frac{k}{n}\right) \right) + f(1) \geqq 0,$$

i. e. $V(n)$ ($\leqq 0$) monotonically increasing.

Thus the inequalities

$$\sum_{x<n\leq(1+\delta)x} V(n)\begin{cases}\leqq V(x+\delta x)\cdot(\delta x+1),\\ \geqq V(x)\cdot(\delta x-1)\end{cases}$$

hold for every positive δ.

On the other hand, by use of (43) we obtain

$$\sum_{x<n\leq(1+\delta)x} V(n)=G(x+\delta x)-G(x)=A\zeta(\alpha)((x+\delta x)^{\alpha}-x^{\alpha})+$$

$$+o((1+\delta)^{\alpha}x^{\alpha})+o(x^{\alpha})=A\zeta(\alpha)((1+\delta)^{\alpha}-1)x^{\alpha}+o(x^{\alpha}),$$

and therefore

$$\left.\begin{matrix}V(x+\delta x)\cdot(\delta x+1)\geqq\\ V(x)\cdot(\delta x-1)\leqq\end{matrix}\right\} A\zeta(\alpha)((1+\delta)^{\alpha}-1)x^{\alpha}+o(x^{\alpha}).$$

Hence

(45) $$V(x)\leqq A\zeta(\alpha)x^{\alpha}\frac{(1+\delta)^{\alpha}-1+o(1)}{\delta x-1}=A\zeta(\alpha)x^{\alpha-1}\frac{(1+\delta)^{\alpha}-1+o(1)}{\delta+o(1)}=$$

$$=A\zeta(\alpha)x^{\alpha-1}\frac{(1+\delta)^{\alpha}-1}{\delta}+o(x^{\alpha-1});$$

we deduce similarly

$$V(x+\delta x)\geqq A\zeta(\alpha)x^{\alpha-1}\frac{(1+\delta)^{\alpha}-1}{\delta}+o(x^{\alpha-1})$$

and, x being replaced by $\frac{1}{1+\delta}$, it follows

(46) $$V(x)\geqq A\zeta(\alpha)x^{\alpha-1}\frac{(1+\delta)^{\alpha}-1}{\delta(1+\delta)^{\alpha-1}}+o(x^{\alpha-1}).$$

According to (45), we have

$$\limsup_{x\to\infty}\frac{V(x)}{x^{\alpha-1}}\leqq A\zeta(\alpha)\frac{(1+\delta)^{\alpha}-1}{\delta};$$

(46) shows that

$$\liminf_{x\to\infty}\frac{V(x)}{x^{\alpha-1}}\geqq A\zeta(\alpha)\frac{(1+\delta)^{\alpha}-1}{\delta(1+\delta)^{\alpha-1}}.$$

Let now $\delta\to+0$, we get

$$\limsup_{x\to\infty}\frac{V(x)}{x^{\alpha-1}}\leqq A\alpha\zeta(\alpha),$$

$$\liminf_{x\to\infty}\frac{V(x)}{x^{\alpha-1}}\geqq A\alpha\zeta(\alpha),$$

consequently

$$\lim_{x\to\infty}\frac{V(x)}{x^{\alpha-1}}=A\alpha\zeta(\alpha),$$

i. e.

(47) $$V(x)\sim A\alpha\zeta(\alpha)x^{\alpha-1}.$$

3⁰. In virtue of (47) (cf. (33))

(48) $$\sum_{k=1}^{[x]} f\left(\frac{k}{[x]}\right)=A\alpha\zeta(\alpha)x^{\alpha-1}+o(x^{\alpha-1}),$$

furthermore

$$\sum_{k=1}^{[x]+1} f\left(\frac{k}{[x]+1}\right)=A\alpha\zeta(\alpha)x^{\alpha-1}+o(x^{\alpha-1}),$$

and so

(49) $$\sum_{k=1}^{[x]+1} f\left(\frac{k}{[x]+1}\right)-\sum_{k=1}^{[x]} f\left(\frac{k}{[x]}\right)=o(x^{\alpha-1}).$$

Since $f(t)$ is non-negative and decreasing in the interval $0<t\leq 1$, we can write

$$0\leq\sum_{k=1}^{[x]} f\left(\frac{k}{[x]}\right)\leq\sum_{k=1}^{[x]} f\left(\frac{k}{x}\right)\leq\sum_{k=1}^{[x]+1} f\left(\frac{k}{[x]+1}\right),$$

so that (cf. (48), (49))

(50) $$\sum_{k=1}^{[x]} f\left(\frac{k}{x}\right)-\sum_{k=1}^{[x]} f\left(\frac{k}{[x]}\right)\leq\sum_{k=1}^{[x]+1} f\left(\frac{k}{[x]+1}\right)-\sum_{k=1}^{[x]} f\left(\frac{k}{[x]}\right)=o(x^{\alpha-1}),$$
$$\sum_{k=1}^{[x]} f\left(\frac{k}{x}\right)=A\alpha\zeta(\alpha)x^{\alpha-1}+o(x^{\alpha-1}).$$

Let $q(t)\equiv f\left(\frac{1}{t}\right)$ for $0<t\leq 1$, then

$$F(x)=\sum_{k=1}^{[x]} f\left(\frac{k}{x}\right)=\sum_{k=1}^{[x]} q\left(\frac{x}{k}\right),$$

which implies, according to Lemma 4,

(51) $$q(x)=\sum_{n=1}^{[x]}\mu(n)F\left(\frac{x}{n}\right).$$

Next we may argue as above (1⁰), by writing

$$q(x)=\sum_{n\leq x}\mu(n)A\alpha\zeta(\alpha)\left(\frac{x}{n}\right)^{\alpha-1}+$$
$$+\sum_{n\leq\frac{x}{\xi}}\mu(n)\left(F\left(\frac{x}{n}\right)-A\alpha\zeta(\alpha)\frac{x^{\alpha-1}}{n^{\alpha-1}}\right)+\sum_{\frac{x}{\xi}<n\leq x}\mu(n)\left(F\left(\frac{x}{n}\right)-A\alpha\zeta(\alpha)\frac{x^{\alpha-1}}{n^{\alpha-1}}\right)$$

and applying (50) We obtain finally ($\alpha > 2$)

$$q(x = A\alpha\zeta(\alpha)x^{\alpha-1}\sum_{n\leq x}\frac{\mu(n)}{n^{\alpha-1}} + o(x^{\alpha-1}),$$

and therefore (cf. (40))

$$q(x) = \frac{A\alpha\zeta(\alpha)}{\zeta(\alpha-1)}x^{\alpha-1} + o(x^{\alpha-1}),$$

$$f\left(\frac{1}{x}\right) \sim \frac{A\alpha\zeta(\alpha)}{\zeta(\alpha-1)}x^{\alpha-1}.$$

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