On factorisable simple groups.

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A group $\mathfrak{G}$ is called factorisable by its subgroups $\mathfrak{H}$ and $\mathfrak{K}$ if each element of $\mathfrak{G}$ may be written in the form $G = HK$ with H in $\mathfrak{H}$, K in $\mathfrak{K}$. The present paper contains a sufficient condition for a factorisable group $\mathfrak{G} = \mathfrak{H}\mathfrak{K}$ to be simple, if the orders of $\mathfrak{H}$ and $\mathfrak{K}$ are relatively prime. Evidently $\mathfrak{H}\mathfrak{K} = \mathfrak{K}\mathfrak{H}$.

In the proof I shall use a result of my earlier paper[1]) which I formulate now as a

L e m m a. *If in the finite group $\mathfrak{G} = \mathfrak{H}\mathfrak{K}$ the orders of the factors $\mathfrak{H}$ and $\mathfrak{K}$ are relatively prime, then every normal subgroup $\overline{\mathfrak{G}}$ of $\mathfrak{G}$ is of the form $\overline{\mathfrak{G}} = \overline{\mathfrak{H}}\overline{\mathfrak{K}}$, where $\overline{\mathfrak{H}}$ and $\overline{\mathfrak{K}}$ are normal subgroups of $\mathfrak{H}$ and $\mathfrak{K}$ respectively.*

I shall now prove the following

T h e o r e m. *If in the factorisable group $\mathfrak{G} = \mathfrak{H}\mathfrak{K}$ the orders of $\mathfrak{H}$ and $\mathfrak{K}$ are relatively prime, if further the groups $\mathfrak{H}$ and $\mathfrak{K}$ are maximal subgroups in $\mathfrak{G}$, then the group $\mathfrak{G}$ is simple, except the case where $\mathfrak{H}$ or $\mathfrak{K}$ is a group of prime order; in the latter case at least one of the factors is a normal subgroup.*

P r o o f. It suffices to show that if $\mathfrak{G}$ is not simple, then either $\mathfrak{H}$ or $\mathfrak{K}$ is not maximal.

a) If the group $\mathfrak{G}$ has a normal subgroup $\overline{\mathfrak{G}}$ and $\overline{\mathfrak{G}} \neq \mathfrak{H}$, $\overline{\mathfrak{G}} \neq \mathfrak{K}$, then by our Lemma $\overline{\mathfrak{G}} = \overline{\mathfrak{H}}\overline{\mathfrak{K}}$ where $\overline{\mathfrak{H}} \subseteqq \mathfrak{H}$, $\overline{\mathfrak{K}} \subseteqq \mathfrak{K}$. Therefore the groups $\mathfrak{H}\overline{\mathfrak{H}}\overline{\mathfrak{K}} = \mathfrak{H}\overline{\mathfrak{K}} = \overline{\mathfrak{K}}\mathfrak{H} = \mathfrak{H}'$ and $\mathfrak{K}\overline{\mathfrak{K}}\overline{\mathfrak{H}} = \mathfrak{K}\overline{\mathfrak{H}} = \overline{\mathfrak{H}}\mathfrak{K} = \mathfrak{K}'$ are subgroups of $\mathfrak{G}$ and at least one of them is proper.

b) If the group $\mathfrak{K}$ is a normal subgroup of $\mathfrak{G}$ and the order of $\mathfrak{H}$ is not a prime, then $\mathfrak{H}$ has an element H such that $\{H\} \subset \mathfrak{H}$, since $\{\mathfrak{K}, H\} \subset \mathfrak{G}$. The proof is the same if $\mathfrak{H}$ is a normal subgroup of $\mathfrak{G}$.

c) If the group $\mathfrak{H}$ is a maximal, but not a normal subgroup and if its order is a prime number, then $\mathfrak{K}$ is necessarily a normal subgroup of $\mathfrak{G}$. In fact, in this case the groups $K^{-1}\mathfrak{H}K$ are different, whenever K runs over all elements of $\mathfrak{K}$ and have only the unit element in common. Thus a known theorem of Frobenius[2]) implies the statement.

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[1]) J. Szép, On the structure of groups which can be represented as the product of two subgroups, *these Acta*, **12 A** (1950), pp. 57—61.

[2]) See e. g. A. Speiser, *Theorie der Gruppen von endlicher Ordnung*, 3 ed. (Berlin, 1937), Theorem 180.