Conservative series to series transformation matrices.

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1. Introduction.

In a recent paper [9][1]) infinite matrices representing *regular* series to series summation methods have been discussed. The present paper is mainly concerned with *conservative* series to series summation matrices, a more general class of matrices. Regular matrices form a subclass of these matrices, and the results obtained in this paper are therefore valid also for this subclass.

The standard method of 'summing' a series of complex terms $u_0+u_1+\cdots$ with partial sums $s_k=u_0+u_1+\cdots+u_k$ is the *transformation of the sequence* s_k *by a matrix of complex elements* $P\equiv(p_{nk})$ *into a convergent sequence* $\sigma_n=\Sigma_k p_{nk}s_k$. The matrix is called *conservative* if the convergence of Σu_k implies that σ_n exists for $n\geqq 0$ and that σ_n tends to a finite limit (which may be different from Σu_k). A conservative sequence to sequence summation matrix is called a *K-matrix* [4, p. 388].

Another method is the *transformation of the series* Σu_k *into a convergent sequence* $\sigma_n=\Sigma_k g_{nk}u_k$ by a matrix $G\equiv(g_{nk})$. A conservative series to sequence summation matrix is called a *β-matrix* [4, 397]. *Sufficient and necessary conditions for G to be a β-matrix are* [4, 394—396]:

(1.1) $\qquad \Sigma_k|g_{nk}-g_{n,k+1}|\leqq M(G)$ for $n=0,1,2,\ldots,$

(1.2) $\qquad g_{nk}$ tends to a finite limit β_k as $n\to\infty$ for $k=0,1,2,\ldots$

We shall employ (1.1) and (1.2) as the *definition* of a β-matrix.

A third method is the *transformation of the series* Σu_k by the matrix $B\equiv(b_{nk})$ *into a convergent series* Σv_n, so that $v_n=\Sigma_k b_{nk}u_k$. If the matrix is *conservative*, i. e. if the convergence of Σu_k implies the existence of v_n for $n=0,1,2,\ldots$ and the convergence of Σv_n (where the two sums may differ), we shall call the matrix B a *δ-matrix*.

The following results will be proved in section 2: A necessary and sufficient condition for a matrix to be a δ-matrix is that the corresponding

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[1]) Numbers in square brackets indicate references given at the end of this paper.

series to sequence method is conservative; the product of a β-matrix and a δ-matrix is a β-matrix; the sum and product of δ-matrices, addition and multiplication being associative; a norm can be defined for δ-matrices under which they form a non-commutative complex Banach algebra with unit element [5, 1.14, p. 12]

In section 3 we discuss the subclass of δ-matrices A which satisfy the condition $a_n = \lim_{k\to\infty} a_{nk} = 0$ for every n, this being necessary for a δ-matrix to be stronger than convergence. Denoting these matrices *δ_0-matrices*, we show that they form a subalgebra of the algebra of δ-matrices under the same norm. The sequence to sequence summation matrix corresponding to a δ_0-matrix is a K-matrix, and the correspondence is an isomorphism. The norm of K-matrices can therefore be introduced to δ_0-matrices. Considering the class $(\mathfrak{B})$ of series with bounded partial sums, if A and B are δ_0-matrices, they apply to every series of $(\mathfrak{B})$ and transform it into a series of $(\mathfrak{B})$, and $A(B(u_k)) = (AB)(u_k)$. A δ_0-matrix and the corresponding K-matrix are equivalent for series of $(\mathfrak{B})$, but examples are given of series of unbounded partial sums and of δ_0-matrices where the two methods are not equivalent.

In section 4 the following δ_0-matrices are studied:

$$\text{(i)} \quad a_{nk} = \binom{k}{n} t^{k-n} t'^{n},$$

$$\text{(ii)} \quad b_{nk} = \binom{n}{k} (1-t) t^{n-k} t'^{k},$$

$$\text{(iii)} \quad c_{nk} = \binom{n+k}{k} (1-t') t^{k} t'^{n}.$$

These matrices are δ_0-matrices when $|t| + |t'| < 1$, they sum the Taylor series $\Sigma u_k z^k$ of the function $f(z)$ in some partial star-domains to the value $f(\alpha z)$, α depending on the matrix. They also display a modified left- or right-translativity. The corresponding series to sequence and sequence to sequence matrices cannot be expressed in simple terms, so that the introduction of δ-matrices was essential. These matrices can be regarded as modified methods of Euler, Taylor and Laurent series continuation discussed in recent papers [3, 6, 9 and 10].

In section 5 special δ_0-matrices are constructed which are efficient for Taylor series at an infinity of isolated points outside its circle of convergence.

2. δ-matrices.

We first consider β-matrices G. The following properties follow from the definition and (1.1), (1.2):

(2.1) *If $\Sigma u_k = s$, then the G-sum of Σu_k is given by $\beta_0 s + \Sigma(\beta_k - \beta_{k+1})(s_k - s)$.*

This is proved in [4, 394—395].

(2.2) *The row limit $\lim_{k\to\infty} g_{nk} = g_n$ exists for every n.*

For $\Sigma_k(g_{nk}-g_{n,k+1})=g_{n0}-\lim_{k\to\infty} g_{nk}$ exists by (1. 1).

(2. 3) $\Sigma_k|\beta_k-\beta_{k+1}|\leq M(G)$.

For $\sum_{k=0}^{r}|\beta_k-\beta_{k+1}|=\lim_{n\to\infty}\sum_{k=0}^{r}|g_{nk}-g_{n,k+1}|\leq M(G)$, for every r.

(2. 4) *$\lim \beta_k=\beta$ exists.*

This follows from (2. 3) as (2. 2) followed from (1. 1).

(2. 5) *The elements of G are bounded.*

For $|g_{nk}|=\left|g_{n0}-\sum_{i=0}^{k-1}(g_{ni}-g_{n,i+1})\right|\leq K(G)+M(G$, where

$$K(G)=\sup_n|g_{n0}|.$$

The column limits β_k, their limit β, and the row limits g_n will be called *the characteristic numbers* of G.

We now consider the summation by series to series transformation of any series $u_0+u_1+\ldots$. Summability by the matrix B means that

(2. 6) $\quad v_n=\Sigma_k b_{nk}u_k$ exists for $n=0, 1, \ldots$, and $\Sigma v_n=s$.

Writing $v_0+v_1+\cdots+v_n=\sigma_n$ and $b_{0k}+b_{1k}+\cdots+b_{nk}=g_{nk}$, we have

(2. 7) $$\sigma_n=\Sigma_k g_{nk}u_k,$$

the existence being implied by (2. 6). Conversely, writing $g_{nk}-g_{n-1,k}=b_{nk}$ $(n=0, 1, \ldots)$ and $g_{-1,k}=0$, we find that the existence of (2. 7) implies the existence of (2. 6). Hence we obtain

Lemma 2. 1. *If the matrices B and G are connected by the relation $g_{nk}=b_{0k}+b_{1k}+\cdots+b_{nk}$, then the series to series transformation by B and the series to sequence transformation by G are equivalent.*

An immediate consequence of the Lemma is:

Theorem 2. 1. *The matrix $B\equiv(b_{nk})$ is a δ-matrix if and only if G, given by $g_{nk}=b_{0k}+b_{1k}+\cdots+b_{nk}$, is a β-matrix.*

We shall call matrices, when they are related as in Theorem 2 1, *corresponding matrices.*

The following properties of β-matrices are easily obtained:

(2. 8) $\Sigma_k|b_{nk}-b_{n,k+1}|\leq 2M(G)$;

(2. 9) $\Sigma_n b_{nk}=\beta_k$ for every k (β_k is defined in (1. 2));

(2. 10) the row limit $\lim_{k\to\infty} b_{nk}=b_n$ exists for every n, and, writing $g_{-1}=0$, $b_n=g_n-g_{n-1}$ (g_n is defined in (2. 2));

(2. 11) $|b_{nk}|\leq 2K(G)+2M(G)$.

It follows from (2. 9) that $b_{nk}\to 0$ as $n\to\infty$ for every k, so that every

δ-matrix is a β-matrix with zero column limits. But not every β-matrix satisfying (2.8) and (2.9) is a δ-matrix, as shown by example (2.10) of [9].

The unit matrix I and the zero matrix O are δ-matrices.

Theorem 2. II. *The product GC of a β-matrix G and a δ-matrix C exists and is a β-matrix.*

Proof By (2.9), $\Sigma_j c_{jk}$ is convergent, and since G is a β-matrix, $(GC)_{nk}=\Sigma_j g_{nj} c_{jk}$ exists for every n and k, hence GC exists. Denoting the product matrix by F, we have

$$\lim_{n\to\infty} f_{nk}=\lim_{n\to\infty}\Sigma_j g_{nj} c_{jk}=G\text{-sum of } \Sigma_j c_{jk} \text{ exists for every } k,$$

so that F satisfies condition (1.2). If H is the β-matrix corresponding to C, with column limits γ_k, we have

$$f_{nk}=\Sigma_j g_{nj}(h_{jk}-h_{j-1,k})=\Sigma_j(g_{nj}-g_{n,j+1})\,h_{jk}+\lim_{j\to\infty} g_{nj}h_{jk}; \tag{2.12}$$

hence

$$f_{nk}-f_{n,k+1}=\Sigma_j(g_{nj}-g_{n,j+1})\,(h_{jk}-h_{j,k+1})+g_n(\gamma_k-\gamma_{k+1}).$$

Using (1.1), (2.5) and (2.3) we obtain

$$\Sigma_k|f_{nk}-f_{n,k+1}|\leq M(G)\,M(H)+\{K(G)+M(G)\}\,M(H), \tag{2.13}$$

so that F satisfies condition (1.1). This concludes the proof.

Corollary 2. II. 1: *The row limits of the product matrix are:*

$$f_n=g_n\gamma+\Sigma_j(g_{nj}-g_{n,j+1})\,h_j.$$

This follows from (2.12), taking the limit of the right-hand side when $k\to\infty$. The series being dominated by $\sup|h_{jk}|\cdot\Sigma_j|g_{nj}-g_{n,j+1}|$, the order of summation and limit can be interchanged.

Corollary 2. II. 2. *The column limits of the product matrix are*

$$\varphi_k=\Sigma_j\beta_j c_{jk}.$$

For $$\varphi_k=\lim_{n\to\infty}\Sigma_j g_{nj}c_{jk}=G\text{-sum-}\Sigma_j c_{jk}=\beta_0\gamma_k+\Sigma_j(\beta_j-\beta_{j+1})(h_{jk}-\gamma_k)=$$
$$=\beta_0\gamma_k+\Sigma_j(h_{jk}-h_{j-1,k})\,\beta_j-\lim_{j\to\infty}\beta_j h_{jk}-\beta_0\gamma_k+\lim_{j\to\infty}\beta_j\gamma_k.$$

Corollary 2. II. 3. $\lim_{k\to\infty}\varphi_k=\beta\gamma+\Sigma_j(\beta_j-\beta_{j+1})\,h_j.$

For the previous corollary gives

$$\varphi_k=\Sigma_j(h_{jk}-h_{j-1,k})\,\beta_j=\Sigma_j(\beta_j-\beta_{j+1})\,h_{jk}+\lim_{j\to\infty}\beta_j h_{jk},$$

the last sum being dominated by $\sup|h_{jk}|\cdot\Sigma_j|\beta_j-\beta_{j-1}|$. Letting $k\to\infty$, we obtain the result.

Corollary 2. II. 4. *The elements of the product matrix are bounded, thus*

$$|f_{nk}|\leq\{M(H)+K(H)\}\,\{2M(G)+K(G)\}.$$

This follows from (2.12), applying (1.1) and (2.5).

Theorem 2. III. *A sufficient and necessary condition for the matrix product GC to exist and be a β-matrix for every β-matrix G is that C should be a δ-matrix.*

The proof is the same as of [9, 2. V]. This result is parallel to a previous result on the product of a T-matrix and a γ-matrix [7], which can be extended to the product of a K-matrix and a β-matrix in a similar way as in theorem 2. II of the present paper.

Theorem 2. IV. *The product of two δ-matrices exists and is a δ-matrix.*

Proof. In the notation of theorem 2. II $F=GC$ is a β-matrix, and the corresponding δ-matrix A is given by

$$a_{nk}=f_{nk}-f_{n-1,k}=\Sigma_j(g_{nj}-g_{n-1,j})\,c_{jk}=\Sigma_j\,b_{nj}c_{jk}, \tag{2. 14}$$

which is the product of the δ-matrices B and C.

Corollary 2. IV. 1. *The row limits of the product matrix are*

$$a_n=b_n\gamma+\Sigma_j(b_{nj}-b_{n,j+1})\,h_j.$$

This follows from corollary 2. II. 1.

Corollary 2. IV. 2. *The elements of the product matrix are bounded, thus*

$$|a_{nk}|\leqq\{2M(H)+2K(H)\}\,\{2M(G)+K(G)\}.$$

This follows from corollary 2. II. IV.

Theorem 2. V. *The product of δ-matrices is associative.*

Proof. Let A, B, C be δ-matrices, F, G, H the corresponding β matrices respectively, $AB=D$, $GC=R$. We consider the double series

$$\Sigma_i\Sigma_j(a_{ni}-a_{n,i+1})\,(g_{ij}-g_{i,j+1})\,h_{jk} \tag{2. 15}$$

which converges absolutely. Summing as indicated, we obtain

$$\begin{aligned}
&\Sigma_i(a_{ni}-a_{n,i+1})\,\{\Sigma_j(h_{jk}-h_{j-1,k})\,g_{ij}-\lim_{j\to\infty} g_{ij}h_{jk}\}\\
&=\Sigma_i(a_{ni}-a_{n,i+1})\,\{\Sigma_j g_{ij}c_{jk}-g_i\gamma_k\}=\Sigma_i(a_{ni}-a_{n,i+1})\,(r_{ik}-g_i\gamma_k)\\
&=\Sigma_i(r_{ik}-r_{i-1,k})\,a_{ni}-\lim_{i\to\infty} a_{ni}r_{ik}-\gamma_k\Sigma_i(a_{ni}-a_{n,i+1})\,g_i\\
&=\Sigma_i\,a_{ni}(BC)_{ik}-a_n\varrho_k-\gamma_k\,\Sigma_i(a_{ni}-a_{n,i+1})\,g_i\\
&=[A(BC)]_{nk}-a_n\varrho_k-\gamma_k(d_n-a_n\beta)\quad\text{by corollary 2. IV. 1.}
\end{aligned}$$

Reversing the order of summation in (2. 15), we obtain

$$\begin{aligned}
&\Sigma_j h_{jk}\{\Sigma_i(g_{ij}-g_{i-1,j}-g_{i,j+1}+g_{i-1,j+1})\,a_{ni}-\lim_{i\to\infty} a_{ni}(g_{ij}-g_{i,j+1})\}\\
&=\Sigma_j h_{jk}\{\Sigma_i a_{ni}(b_{ij}-b_{i,j+1})-a_n(\beta_j-\beta_{j+1})\}\\
&=\Sigma_j h_{jk}(d_{nj}-d_{n,j+1})-a_n\,\Sigma_j(\beta_j-\beta_{j+1})\,h_{jk}\\
&=\Sigma_j(h_{jk}-h_{j-1,k})\,d_{nj}-\lim_{j\to\infty} d_{nj}h_{jk}-a_n\,\Sigma_j\beta_j c_{jk}+a_n\lim_{j\to\infty}\beta_j h_{jk}\\
&=\Sigma_j d_{nj}c_{jk}-d_n\gamma_k-a_n\varrho_k+a_n\gamma_k=[(AB)\,C]_{nk}-a_n\varrho_k-\gamma_k(d_n-a_n\beta),
\end{aligned}$$

showing that $[A(BC)]=[(AB)C]$.

Corollary 2. V. 1. *The product of a β-matrix on the left followed by δ-matrices is associative.*

This follows from the fact that if (a_{nk}) is a δ-matrix, so is $(a_{n-1,k})$, i. e. if a zero row is added to A. Hence all the double sums $\Sigma_i\Sigma_j a_{hi}b_{ij}c_{jk}$ for $h=0, 1, \dots, n$ can be inverted as well as their sum $\Sigma_i\Sigma_j f_{ni}b_{ij}c_{jk}$.

Theorem 2. VI. *Every finite linear combination of δ-matrices is a δ-matrix.*

Proof. If A and B are δ-matrices, F, G the corresponding β-matrices, x, y complex numbers, $xF+yG=H$, then $|h_{nk}-h_{n,k+1}|\leqq|x(f_{nk}-f_{n,k+1})|+ +|y(g_{nk}-g_{n,k+1})|$, so that

$$\Sigma_k|h_{nk}-h_{n,k+1}|\leqq|x|M(F)+|y|M(G), \tag{2.16}$$

and $\lim h_{nk}$ exists. Thus H is a β-matrix and $C=xA+yB$ a δ-matrix.

It follows from the last three theorems that δ-matrices form a non-commutative ring, with the unit matrix as unit element and the zero matrix as zero element. The ring contains *zero-divisors*, for example $AB=0$ when $a_{0k}=1$ for all k, all the other elements of A being zero, $b_{0k}=-1$, $b_{1k}=1$, all the other elements of B being zero.

It is possible to define a *norm* for δ-matrices in the following way: If B is a δ-matrix, G the corresponding β-matrix, $N(G)=\sup_n\Sigma_k|g_{nk}-g_{n,k+1}|$, $K(G)=\sup_n g_{n0}$ (as in (2.5)), the norm of B is defined as the number

$$\|B\|=2\{N(G)+K(G)\}. \tag{2.17}$$

The following properties of the norm are easily verified:

(2.18) (i) $\|B\|\geqq 0$ and $\|B\|=0$ if and only if B is the zero matrix.
(ii) $\|xB\|=|x|\,\|B\|$, x being a complex number.
(iii) $\|B+C\|\leqq\|B\|+\|C\|$.
(iv) $\|BC\|\leqq\|B\|\cdot\|C\|$.
(v) $|b_{nk}|\leqq\|B\|$.

Here (i), (ii) follow from the definition of the norm, (iii) from (2.16), (v) from (2.11). To prove (iv), we use the notation in the proof of theorem 2. V. It follows from (2.13) that $N(F)\leqq N(H)\{2N(G)+K(G)\}$. Also $f_{n0}=\Sigma_j g_{nj}c_{j0}= =\Sigma_j(g_{nj}-g_{n,j+1})h_{j0}+\lim_{j\to\infty} g_{nj}h_{j0}$, hence $K(F)\leqq K(H)\{2N(G)+K(G)\}$, and (iv) follows by adding $N(F)$ and $K(F)$.

Theorem 2. VII. *With the given norm, δ-matrices form a non-commutative complex Banach algebra with unit element.* [5, 1.14, p. 2; 1.11, p. 10].

Proof. It is sufficient to prove that the space of δ-matrices is *complete* in the topology induced by the metric $d(B, C)=\|B-C\|$, i. e. that every Cauchy sequence $A(t)$ $(t=1, 2, \dots)$ of δ-matrices converges to a limit matrix A which is a δ-matrix. We first prove

Lemma 2. VII. 1. *If $\Sigma_t\|A(t)\|$ converges, then $\Sigma_t A(t)=A$ exists and is a δ-matrix.*

Proof. Let $F(t)$ be the β-matrix corresponding to $A(t)$. The sums $\Sigma_k|\Sigma_t\{f(t)_{nk}-f(t)_{n,k+1}\}|\leq\Sigma_t\Sigma_k|f(t)_{nk}-f(t)_{n,k+1}|$, $\Sigma_t f(t)_{nk}$ and $\lim\limits_{n\to\infty}\Sigma_t f(t)_{nk}=$ $=\Sigma_t\lim\limits_{n\to\infty}f(t)_{nk}$ are all dominated by the series $\Sigma_t\|A(t)\|$, hence the matrix $F=\Sigma_t F(t)$ exists and is a β-matrix, and $A=\Sigma_t A(t)$ is the corresponding δ-matrix.

Corollary to the lemma. *The characteristic numbers of the sum matrix are the sums of the corresponding characteristic numbers.*

This follows from the uniform convergence of all the series concerned.

Lemma 2. VII. 2. *The space of δ-matrices is complete under the given norm.*

Proof. Assuming that $A(t)$ is a Cauchy sequence, so that for $t,t'>T(\varepsilon)$, $\|A(t)-A(t')\|<\varepsilon$, we find that $|a(t)_{nk}-a(t')_{nk}|<\varepsilon$, hence $a(t)_{nk}\to a_{nk}$ as $t\to\infty$. Thus $A(t)$ tends in each element to a matrix $A=(a_{nk})$. Determining a sequence of positive integers $t_j>t_{j-1}$ such that $\|A(t_j)-A(t_{j-1})\|<2^{-j}$, so that the series $\Sigma_j\|A(t_j)-A(t_{j-1})\|$ converges, we find that, by the previous lemma, the series $A(t_1)+\sum\limits_{j=2}^{\infty}\{A(t_j)-A(t_{j-1})\}$ converges, and its sum $\lim\limits_{j\to\infty}A(t_j)=\lim\limits_{t\to\infty}A(t)$ is a δ matrix. Since, for $t>T(\varepsilon)$, $\|A(t)-A\|\leq\varepsilon$, the convergence is strong. This proves the lemma, and therefore the theorem.

Corollary 2. VII. 1. *The norm of the limit matrix is the limit of the norms*

For $\big|\|A(t)\|-\|A(t')\|\big|\leq\|A(t)-A(t')\|<\varepsilon$, and hence $\big|\|A(t)\|-\|A\|\big|\leq$ $\leq\|A(t)-A\|\leq\varepsilon$.

Corollary 2. VII. 2. *The characteristic numbers of the limit matrix are the limits of the corresponding characteristic numbers.*

This follows from the corollary to lemma 2. VII. 1.

It is known that in a Banach algebra with unit element the neighbourhood of the unit element consists of regular elements. If A is a δ-matrix such that $\|A-I\|<1$, the reciprocal A^{-1} is the power series $I+(I-A)+(I-A)^2+\ldots$, [5, Theorem 5.2.1, p. 92]. More generally if $\|A-\lambda I\|<|\lambda|$, A has a two-sided δ-matrix reciprocal, given by $\lambda^{-1}\{I+(I-A/\lambda)+(I-A/\lambda)^2+\ldots\}$.

If all the column-sums β_k of a δ matrix B are equal to unity, the matrix is a regular series to series summation matrix, called an α-matrix [9, 541]. The α-matrices form a subclass of the algebra of δ-matrices which is not an algebra since the sum of two α-matrices is not an α-matrix. But if A is an α-matrix and $\|A-I\|<1$, then A^{-1} obtained as the sum of a convergent power series is an α-matrix, since all the column-sums of the matrix $I-A$ are zero, and therefore, by Corollary 2. II. 2, so are the column-sums of $(I-A)^j$.

The same can be proved when $\|A-\lambda I\|<|\lambda|$, when the column-sums of the reciprocal are given by $\lambda^{-1}\Sigma(1-1/\lambda)^j$, giving unity.

A column vector $u_0, u_1, \ldots$ can be regarded as an infinite matrix with zero columns except the first column. It is easy to verify that such a matrix U is a δ-matrix if and only if Σu_k is convergent. In this sense, then *convergent series are elements of the algebra*. This gives the following theorem:

Theorem 2. VIII. *If the α-matrix A has a right reciprocal A^{-1} which is a δ-matrix, then A^{-1} is an α-matrix.*

Proof. If U is a column vector such that Σu_k is convergent, $V=A^{-1}U$ is a column vector such that Σv_n is convergent, since A^{-1} is a δ-matrix. Hence $AV=AA^{-1}U=U$, and A being an α-matrix, it follows that $\Sigma v_n=\Sigma u_k$. Thus A^{-1} sums every convergent series to its sum, and is therefore an α-matrix.

3. δ_0-matrices.

The class of δ-matrices so far considered is the class of matrices which transform convergent series into convergent series. Their use for generalized summation of series however requires more: they should transform at least one divergent series into a convergent series. In other words: they should be *'stronger than convergence'*. It is possible, at this stage, to exclude a wide class of trivial δ-matrices from further investigations by the following result:

Theorem 3. I. *A necessary condition that a δ-matrix $B\equiv(b_{nk})$ should be stronger than convergence is that*

$$b_n=\lim_{k\to\infty} b_{nk}=0 \quad \textit{for} \quad n=0,1,2,\ldots. \tag{3. 1}$$

The proof is based on the following lemma:

Lemma 3. I. *If $\Sigma_k|b_k-b_{k+1}|=M<\infty$, and* $\lim b_k\neq 0$, *then the convergence of $\Sigma b_k u_k$ implies the convergence of Σu_k.*

Proof of the lemma: By hypothesis there exist positive numbers r and R such that for $k\geq r$, $|b_k|\geq R$. Hence $\sum_r^\infty |1/b_k-1/b_{k+1}|\leq M/R^2$, and by a lemma due to ABEL and HADAMARD [**4**, 394], $\Sigma c_k/b_k$ converges whenever Σc_k converges. The lemma is proved by taking $c_k=b_k u_k$.

The theorem then follows from (2. 8) and (2. 10).

Corollary. *A necessary condition that a β-matrix $G\equiv(g_{nk})$ be stronger than convergence is that*

$$g_n=\lim_{k\to\infty} g_{nk}=0 \quad \textit{for} \quad n=0,1,2,\ldots. \tag{3. 2}$$

That the condition (3. 1) is not sufficient, is shown by the unit matrix.

We shall call a β-matrix satisfying (3. 2) a *β_0-matrix*, and a δ-matrix satisfying (3. 1) a *δ_0-matrix*. Obviously conditions (3. 1) and (3. 2) are equivalent: *If B is a δ_0-matrix, the corresponding matrix G is a β_0-matrix, and conversely.*

Theorem 3. II. *Under the norm (2. 17) δ_0-matrices form a non-commutative complex Banach algebra with unit element*

Proof. The sum of two δ_0-matrices is obviously a δ_0-matrix. That the product is a δ_0-matrix follows from corollary 2. IV. 1 since $b_n=0$, $h_j=0$ implies $a_n=0$. The unit matrix and the zero matrix are δ_0-matrices. Theorem 2. VII, the two lemmas and the corollaries apply to δ_0-matrices, and show in particalar that the space is complete. This concludes the proof.

The column vectors U, regarded as infinite matrices, are δ_0-matrices if and only if Σu_k is convergent. In this sense, convergent series are elements of the algebra of δ_0-matrices.

A significant property of δ_0-matrices is revealed by investigating the corresponding sequence to sequence summation matrices. Such a matrix P is conservative, and called a K matrix, and is defined by [4, 385]

(3. 3) $\quad \Sigma_k |p_{nk}| \leq M(P)$ for $n=0,1,2,\ldots,$

(3. 4) $\quad p_{nk}$ tends to a finite limit π_k as $n\to\infty$ for $k=0,1,2,\ldots,$

(3. 5) $\quad \Sigma_k p_{nk}=p_n$ tends to a finite limit p as $n\to\infty$.

The corresponding series to sequence summation matrix G is given by

(3. 6) $$g_{nk}=p_{nk}+p_{n,k+1}+\cdots.$$

It is known that if P is a K-matrix, G is a β-matrix, and in fact it is a β_0-matrix. Conversely, if G is given, P is given by

(3. 7) $$p_{nk}=g_{nk}-g_{n,k+1}.$$

If G is a β-matrix, P given by (3. 7) is not necessarily a K-matrix [4, 399]. The correspondence between G and P, defined by (3. 7), is not one-to-one, for P is unaltered when g_{nk} is replaced by $g_{nk}+g'_n$, where g'_n is an arbitrary sequence. Starting with G, and using (3. 7) and (3. 6) in turn, the correspondence $G\to P\to G'$ gives

$$g'_{nk}=p_{nk}+p_{n,k+1}+\cdots=g_{nk}-g_{n,k+1}+g_{n,k+1}-g_{n,k+2}+\cdots=g_{nk}-g_n$$

and $G=G'$ if and only if $g_n=0$. Using the one-one correspondence between δ_0- and β_0-matrices, we obtain

Theorem 3. III. *There is a one-to-one correspondence between δ_0-matrices B, and K-matrices P, expressed by either of the equivalent formulae*

(3. 8) $$p_{nk}=\sum_{j=0}^{n}(b_{jk}-b_{j,k+1}),$$

(3. 9) $$b_{nk}=\sum_{j=k}^{\infty}(p_{nj}-p_{n-1,j}) \quad \textit{with } p_{-1,j}=0.$$

Theorem 3. IV. *The correspondence in theorem 3. III is an isomorphism.*

Proof. If B, C are δ_0-matrices, G, H the corresponding β_0-matrices and P, Q the corresponding K-matrices, it follows from (3. 8) and (3. 9) that $B+C$ corresponds to $P+Q$ Again

$$(PH)_{nk} = \Sigma_j p_{nj} h_{jk} = \Sigma_j (g_{nj} - g_{n,j+1}) h_{jk} = \Sigma_j g_{nj} (h_{jk} - h_{j-1,k}),$$

since $g_{nj} h_{jk} \to 0$ as $j \to \infty$. Thus $(PH)_{nk} = (GC)_{nk}$, and $PH = GC$ is a β_0-matrix, PQ is the corresponding K-matrix, and BC the corresponding δ_0-matrix. Thus PQ corresponds to BC. This concludes the proof of the theorem.

We can now introduce a new norm for δ_0-matrices. It is known that K-matrices form a Banach algebra under the norm

$$\|P\| = \sup_n \Sigma_k |p_{nk}|, \tag{3. 10}$$

and, using the last theorem, we can define the norm of a δ_0-matrix B as the norm of the corresponding K-matrix P, i. e.

$$\|B\|_K = \|P\|. \tag{3. 11}$$

The unit matrix is a K-matrix and the corresponding δ_0-matrix is the same matrix. In the same way, the zero matrix corresponds to itself. We have therefore $\|I\|_K = \|I\| = 1$, whereas the δ-norm of I as defined in (2. 17) has the value 4. The norm (3. 11) for δ_0-matrices has the properties (2. 18) including the property (v) (which is not generally required in abstract algebras, but is essential for infinite matrices to establish completeness of the space).

We consider now series Σu_k with partial sums s_k, and use the following notations. We denote a sequence to sequence transformation by $P, Q, \ldots$:

$$\sigma_n = P(s_k) = \Sigma_k p_{nk} s_k;$$

a series to sequence transformation by $G, H, \ldots$:

$$\sigma_n = G(\Sigma u_k) = \Sigma_k g_{nk} u_k;$$

a series to series transformation by $B, C, \ldots$:

$$\Sigma v_n = B(\Sigma u_k) = \Sigma_n (\Sigma_k b_{nk} u_k).$$

We denote the class of series Σu_k, such that the *partial sums s_k are bounded*, by $(\mathfrak{B})$. It is known that a K-matrix P transforms every bounded sequence s_k into a bounded sequence σ_n. If G is the corresponding β_0-matrix, we have

$$P(s_k) = G(\Sigma u_k) \quad [4, 398-399]. \tag{3. 12}$$

Hence we obtain

Theorem 3. V. *Every δ_0-matrix B transforms every series Σu_k of $(\mathfrak{B})$ into a series Σv_n of $(\mathfrak{B})$.*

We also have

Theorem 3. VI. *If B and C are δ_0-matrices, and the series Σu_k belongs to $(\mathfrak{B})$ then $B[C(\Sigma u_k)] = (BC)(\Sigma u_k)$.*

Proof. If G, H are the corresponding β_0-matrices, P, Q the corresponding K-matrices, s_k the partial sums of Σu_k, we have $P[Q(s_k)] = (PQ)(s_k)$. Now

$Q(s_k) = H(\Sigma u_k) = \sigma_n$, and if $\sigma_n - \sigma_{n-1} = v_n$, $P[Q(s_k)] = P(\sigma_n) = B(\Sigma v_n) = B[C(\Sigma u_n)]$. Again, $(PQ)(s_k) = (BC)(\Sigma u_k)$. This proves the theorem.

It follows from the identity (3. 12) that the sequence to sequence and series to series summation methods are identical for all series of the class $(\mathfrak{B})$, That this is not the case for series with unbounded partial sums, is shown by the following examples:

(3. 13) $$u_k = (k+1)^2,\ b_{nk} = (-1)^k/(n+1)(n+2)(k+1)^3.$$

Here $g_{nk} = (-1)^k(n+1)/(n+2)(k+1)^3$ so that $\Sigma_k|g_{nk} - g_{n,k+1}| < 2\Sigma(k+1)^{-3}$, and g_{nk} tends to a limit as $n \to \infty$, and to zero as $k \to \infty$. Hence B is a δ_0-matrix, and

$$\Sigma v_n = \Sigma_n \Sigma_k b_{nk} u_k = \Sigma_n \Sigma_k (-1)^k/(k+1)(n+1)(n+2) = \log 2,$$

so that the B-sum exists. Again $p_{nk} = g_{nk} - g_{n,k+1} = O(k^{-3})$, $s_k = O(k^3)$, hence $\Sigma_k p_{nk} s_k$ diverges, so that the P-sum does not exist.

(3. 14) $$u_k = (-1)^k(2k+1),\ b_{nk} = 1/(n+1)(n+2)(k+1).$$

Here $\Sigma_k b_{nk} u_k$ diverges, so that the B-sum does not exist. Again

$$s_k = (-1)^k(k+1),\ g_{nk} = (n+1)/(n+2)(k+1),$$
$$p_{nk} = (n+1)/(n+2)(k+1)(k+2)$$

and $\Sigma_k p_{nk} s_k = \Sigma_k (-1)^k (n+1)/(k+2)(n+2) \to 1 - \log 2$ as $n \to \infty$. Hence the P-sum exists.

4. Some examples of δ_0-matrices with applications to Taylor series.

When a δ_0-matrix B is applied to the Taylor series $\Sigma u_k z^k$ representing the function $f(z)$ in its circle on convergence, it cannot be expected that the generalized sum $S(z)$ shall be the 'right' value $f(z)$. The relation between the two values *for convergent series* with partial sums $s_k(z) \to f(z)$ can be expressed, using (2. 1), as $S(z) = \beta_0 f(z) + \Sigma(\beta_k - \beta_{k+1})\{s_k(z) - f(z)\}$, and the generalized sum is the analytic continuation of $S(z)$ in an open connected domain of summability containing the circle of convergence.

In this section we consider matrices for which the relation $f(z) \to S(z)$ is the simplest possible: $S(z) \equiv f(\alpha z)$, α being a complex number depending on the matrix, but independent of the series to which it is applied. Such matrices turn up as a natural generalization of matrices discussed by the author in a previous paper [**9**, sections 3, 4, 5]. The corresponding sequence to sequence summation matrices have recently also been discussed by other writers [**3** and **6**]. We restrict our attention to series to series methods, the corresponding sequence to sequence matrices having too complicated expressions to be of any use.

As in the case of regular summation methods, the domain of summability for general Taylor series can be defined, if the domain of summability $D(B)$ for the series Σz^k is known. For conservative methods we require a modified restatement of a theorem due to E. BOREL [**1**, 197—200], which we give here without proof:

Theorem 4. I. *Let $f(z)$ be represented by the series $\Sigma u_k z^k$ in its principal star-domain. If $\varphi_n(z)=\sum_{j=0}^{n}\sum_{k=0}^{\infty} b_{nk} z^k \to \varphi(z)$ as $n\to\infty$ uniformly in every closed region of a star-domain $D(B)$, then the B-sum of $\Sigma u_k z^k=\Sigma_n \Sigma_k b_{nk} u_k z^k$ exists in the partial star-domain corresponding to $D(B)$, and its value is $\int_\Gamma w^{-1} f(w)\,\varphi(z/w)\,dw$, Γ being a small circle about the origin, inside and on which $f(w)$ is regular.* (Star-domains are defined in [**9**, 3. 23, p. 551]).

In the particular case when $\varphi(z)\equiv 1/(1-\alpha z)$, the B-sum is

$$\int_\Gamma f(w)/(w-\alpha z)\,dw=f(\alpha z).$$

A natural generalization of the Taylor series continuation method [**9**, section 3; **3** and **6** the method $T(\alpha)$], is the matrix $A(t, t')$ depending on the two complex parameters t and t', given by

$$a_{nk}=a(t,t')_{nk}=\binom{k}{n} t^{k-n}\, t'^{n} \qquad (n=0,1,2,\ldots). \tag{4. 1}$$

Applying $A(t, t')$ to the series Σz^k, we obtain

$$v_n(z)\equiv \Sigma_k a_{nk} z^k=(t'/t)^n \sum_{k=n}^{\infty}\binom{k}{n}(tz)^k,$$

$$\Sigma v_n(z)=(1-tz)^{-1}\Sigma(t'z)^n/(1-tz)^n=1/(1-\alpha z)\quad (\alpha=t+t');$$

provided that

$$\text{(i)}\ \left|\frac{1}{z}\right|>|t|,\quad \text{(ii)}\ \left|\frac{1}{z}-t\right|>|t'|, \tag{4. 2}$$

which defines the domain $D(A)$.

$D(A)$ contains the unit circle in its inside, hence

$$A(t,t') \text{ is a } \delta_0\text{-matrix when } |t|+|t'|<1, \tag{4. 3}$$

and we shall consider this case only.

Assuming (4. 3), the following types of summability domains may occur, (i) being always the inside of the circle with centre 0, radius $1/|t|$:

(a) $|t'|\geq|2t|$; $D(A)$ is the inside of the circle (ii) which is inside the circle (i);

(b) $|2t|>|t'|>|t|$, the circles (i) and (ii) intersect, and $D(A)$ is inside both circles;

(c) $|t'|=|t|$, $D(A)$ is the larger segment of circle (i) cut of by the line (ii) which is the perpendicular bisector of the radius joining the origin to the point $z=1/t$;

(d) $|t'|<|t|$, the two circles intersect, and $D(A)$ is inside (i) and outside (ii).

In all cases $D(a)$ depends on t and $|t'|$, but not on $\arg(t')$. The union of all domains $D(t,t')$ for t,t' satisfying (4.3), is the whole open z-plane.

The question also arises, whether $A(t,t')$ is *'relatively conservative'*, i. e. whether it contains all points z such that αz is inside the circle of convergence. This requires that $D(A)$ contain the circle $|z|<1/|\alpha|$, which is satisfied if and only if $|\alpha|\geq|t|+|t'|$, and since $\alpha=t+t'$, we must have $\arg(t)=\arg(t')$.

The method $A(t,t')$ has restricted translative properties [e. g. **9,** 3.11, 547]. Writing, if they exist, $v_n=\Sigma_k a_{nk}u_k$, $v'_n=\Sigma_k a_{n,k+1}u_k$, $\sigma_n=v_0+v_1+\ldots+v_n$, $\sigma'_n=v'_0+v'_1+\ldots+v'_n$, we have $v'_n-tv_n=t'v_{n-1}$ for $n=1,2,\ldots$, $v'_0-tv_0=0$, hence $\sigma'_n=t\sigma_n+t'\sigma_{n-1}$, so that when $\sigma_n\to S$, $\sigma'_n\to\alpha S$. Thus:

"*$A(t,t')$ sums the series $u_0+u_1+\ldots$ to the value S" implies that it sums the series $0+u_0+u_1+\ldots$ to the value αS.* We may say that $A(t,t')$ *is translative to the left with factor α.*

The generalization of the Euler matrix on similar lines gives the matrix $B(t,t')$ as the transpose of $A(t',t)$ multiplied by $(1-t')$, i. e.

(4.4) $$b_{nk}=b(t,t')_{nk}=\binom{n}{k}(1-t')\,t'^{\,n-k}t^k.$$

Applying $B(t,t')$ to the series Σz^k, we obtain

$$v_n(z)=(1-t')\,\Sigma_k\binom{n}{k}t'^{\,n-k}(tz)^k=(1-t')\,(t'+tz)^n,$$

$$\Sigma v_n(z)=(1-t')/(1-t'-tz)=1/(1-\alpha z)\qquad(\alpha=t/(1-t')),$$

provided that

(4.5) $$|z+t'/t|<1/|t|,$$

defining $D(B)$ which is a circle. This contains the unit circle in its interior, i. e. *$B(t,t')$ is a δ_0-matrix when* $|t|+|t'|<1$. The union of all domains $D(B)$ is the whole open z-plane.

The matrix is in addition *'relatively conservative'* if $D(B)$ contains the circle $|z|<1/|\alpha|$, i. e. if $|1-t'|\leq 1-|t'|$; this requires that *t' be positive.*

Translative properties of $B(t,t')$ follow from the identity $t\sigma_n=\sigma'_{n+1}-t'\sigma'_n$, which can be established in the same way as for $A(t,t')$. When $\sigma'_n\to S'$, $\sigma_n\to S'(1-t')/t=S'/\alpha$. Hence *$B(t,t')$ is translative to the right with factor* $1/\alpha$.

The generalization of the Laurent summation method [**9,** section 5] and [**6,** the method $S(\alpha)$] on similar lines gives the matrix $C(t,t')$ defined by

(4.6) $$c_{nk}=c(t,t')_{nk}=\binom{n+k}{k}(1-t')\,t^k t'^{\,n}.$$

Applying $C(t,t')$ to the series Σz^k, we obtain

$$c_n(z) = (1-t')t'^n \sum_k \binom{n+k}{k}(tz)^k,$$

$$\sum c_n(z) = (1-t')(1-tz)^{-1}\sum_n t'^n/(1-tz)^n = 1/(1-\alpha z) \qquad (\alpha = t/(1-t'))$$

provided that

(4. 7) $\quad$ (i) $|z| < 1/|t|$ $\quad$ (ii) $|z - 1/t| > |t'/t|$

This defines the domain $D(C)$ inside the circle (i) and outside the circle (ii). $D(C)$ contains the unit circle in its inside, i. e. *$C(t, t')$ is a δ_0-matrix when* $|t|+|t'|=1$. The union of all domains $D(C)$ is the whole open z-plane.

The method is in addition *'relatively conservative'* if $D(C)$ contains the circle $|z| < 1/|\alpha|$, i. e. $|1-t'| \leq 1-|t'|$. This requires that *t' be positive.*

The translative properties of $C(t, t')$ follow from the identity $t\sigma_n = \sigma'_n - t'\sigma'_{n-1}$, so that when $\sigma'_n \to S'$, then $\sigma_n \to S'(1-t')/t = S'/\alpha$. Hence *$C(t, t')$ is translative to the right with factor $1/\alpha$.*

5. δ_0-matrices, efficient at a countable infinity of isolated points.

Regular sequence to sequence summation methods which are efficient at isolated points have been given in [**2,** 53—55], and later extended to a finite number of points for series to sequence methods in [**8,** section 6, 11—13]. A further extension of these results is possible by constructing δ_0-matrices as elements of an abelian multiplicative group of infinite matrices.

The group is generated by the unit matrix I and the *diagonal vector*

$$E \equiv \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ . & . & . & . & \dots \end{pmatrix}$$

The matrix A is given by

(5. 1) $$A = I + e_1 E + e_2 E^2 + \dots,$$

where the e_j are complex numbers, satisfying the following condition:

(5. 2) *the function* $\varphi(w) = 1 + e_1 w + e_2 w^2 + \dots$ *is regular and* $\neq 0$ *for* $|w| \leq 1$.

It follows from (5. 2) that $\sum|e_j| < \infty$, and that if $1/\varphi(w) = 1 + d_1 w + d_2 w^2 + \dots$, then $\sum|d_j| < \infty$.

The reciprocal of A is the matrix

(5. 3) $$A^{-1} = I + d_1 E + d_2 E^2 + \dots.$$

Both A and A^{-1} are δ_0-matrices, with $\|A\|_K \leq 2 + \sum|2e_j|$ and $\|A^{-1}\| \leq 2 + \sum|2d_j|$.

Theorem 5. 1. *The δ_0-matrix A, given by* (5. 1) *and* (5. 2), *is inefficient for all divergent series with bounded partial sums.*

Proof. If $\sum u_k$ is such a series, and if A is efficient for this series we have $\Sigma v_n = A(\Sigma u_k)$ is convergent, hence so is $A^{-1}(\Sigma v_n) = A^{-1}[A(\Sigma u_k)]$. But, by theorem 3. VI, the last expression is equal to $(A^{-1}A)(\Sigma u_k) = \Sigma u_k$, which is divergent.

Applying A to the series Σz^k, we obtain

$$v_n(z) = z^n(1 + e_1 z + e_2 z^2 + \ldots) = z^n \varphi(z),\ \Sigma v_n(z) = \varphi(z)/(1-z),$$

both series being convergent inside the unit circle, and, trivially, at those zeros of $\varphi(z)$ which are inside (or possibly on) its circle of convergence.

If, for example, $\varphi(w)$ is an integral function with an infinity of zeros $w_1, w_2, \ldots$ outside the unit circle, then A sums the series Σz^k to the value $\varphi(z)/(1-z)$ inside the unit circle, and to the value zero at the isolated points $z = w_1, w_2, \ldots$. A suitable function for construction is for example $f(w) \equiv \cos w$, and the corresponding matrix is the matrix

$$A = I - E^2/2! + E^4/4! - \ldots = \begin{pmatrix} 1 & 0 & -1/2! & 0 & 1/4! & \ldots \\ 0 & 1 & 0 & -1/2! & 0 & \ldots \\ 0 & 0 & 1 & 0 & -1/2! & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \ldots \end{pmatrix}.$$

The behaviour of the matrix A (defined in (5. 1)), for other series than Σz^k outside the circle of convergence cannot be deducted from theorem 4. I. For each Taylor series the set of isolated points of summability may differ. For example if A is applied to the binomial series $\Sigma_k \binom{p+k}{k} z^k$ $(p=2,3,\ldots)$, we obtain

$$v_n(z) \equiv \Sigma_k a_{nk} \binom{p+k}{k} z^k = (z^n/p!)\, d^p [z^p \varphi(z)]/dz^p,$$

so that A sums the series at points inside the circle of convergence to the value $[(1-z)p!]^{-1} d^p [z^p \varphi(z)]/dz^p$, and to the value 0 at those isolated zeros of the function $\varphi_p(z) \equiv d^p [z^p \varphi(z)]/dz^p$ which are inside (or possibly on) the of convergence of $\varphi(z)$.

I wish to express my thanks to Professor P. DIENES for suggesting theorems 2. I—IV. I am also indebted to Dr. L. S. BOSANQUET, who gave the proof of lemma 3. I, and to Dr. R. E. EDWARDS, wo suggested lemma 2. VII. 2 and simplified my original proof.

Added in proof (*March 12, 1951*): MEYER–KÖNIG [6, p. 257] remarks that the Taylor summability method was introduced by G. H. HARDY and J. E. LITTLEWOOD as 'circle method' in a paper in the *Rendiconti Circolo Mat. Palermo*, **41** (1916), pp. 36—53. I found recently in the Hungarian textbook of M. BEKE: *Differentiál és Integrálszámitás*, vol. 2 (Budapest, 1916), pp. 433—435 another interesting way of considering analytic continuation as a generalized mean by sequence to sequence transformation. BEKE remarks that this method has been communicated to him verbally by M. FEKETE. FEKETE obtains the summability matrix in the following way:

If $f(z)=\Sigma u_k z^k$ has a radius of convergence R, and $0<|\alpha|<|z|<R$, α/z being positive, then the series can be continued to

$$\Sigma f^{(n)}(\alpha)(z-\alpha)^n/n!.$$

Denoting the partial sums of the two series by $s_k(z)=u_0+u_1 z+\ldots+u_k z^k$ and

$$\sigma_n(z)=f(\alpha)+f'(\alpha)(z-\alpha)+\ldots+f^{(n)}(\alpha)(z-\alpha)^n/n!,$$

we see that

$$\sigma_n(z)=\{(z-\alpha)^{n+1}/n!\}\, d^n\{g(\alpha)\}/d\alpha^n, \text{ where } g(\alpha)\equiv f(\alpha)/(z-\alpha).$$

But

$$g(\alpha)=(\Sigma u_k \alpha^k)(\Sigma \alpha^k/z^{k+1})=\Sigma \alpha^k s_k(z)/z^{k+1},$$

hence

$$\sigma_n(z)=(1-\alpha/z)^{n+1}\Sigma\binom{k}{n}(\alpha/z)^{k-n}s_k(z),$$

which is the transform of $s_k(z)$ by the upper semi-matrix

$$a_{nk}=\binom{k}{n}(\alpha/z)^{k-n}(1-\alpha/z)^{n+1}.$$

Since $a_{nk}\geqq 0$ and $\Sigma_k a_{nk}=1$, the method is a regular sequence to sequence summation method.

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