Remarks on factorizable groups.

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Recently J. SZÉP[1]) obtained some results on factorizable groups. His results permit a slight and easily provable generalization which we wish to remark in this note.

(I) *Let* $\mathfrak{G}$ *be a factorizable group such that*

$$\mathfrak{G}=\mathfrak{S}\cdot\mathfrak{A}$$

where $\mathfrak{S}$ *is nilpotent and* $\mathfrak{A}$ *abelian. Then* $\mathfrak{G}$ *is solvable.*

Proof. An induction argument can be used with respect to the order of $\mathfrak{G}$; thus we may assume that every proper factor group of $\mathfrak{G}$ is solvable, and we have only to prove the existence of a solvable normal subgroup of $\mathfrak{G}$.

We can suppose that $\mathfrak{S}$ is a maximal subgroup of $\mathfrak{G}$. In fact, if a proper subgroup $\mathfrak{H}$ of $\mathfrak{G}$ contains $\mathfrak{S}$ properly, $\mathfrak{H}=\mathfrak{S}\cdot\mathfrak{A}\cap\mathfrak{H}$. Therefore $\mathfrak{H}$ is solvable by our induction hypothesis. Further $\mathfrak{H}$ contains a normal subgroup $\mathfrak{N}$ of $\mathfrak{G}$. which is seen by a result of J. SZÉP and L. RÉDEI[2]). Since $\mathfrak{H}$ is solvable and $\mathfrak{H}\supset\mathfrak{N}$, $\mathfrak{N}$ is solvable. Then $\mathfrak{G}$ is solvable.

Further we san suppose that $\mathfrak{S}$ and $\mathfrak{A}$ have relatively prime orders. In fact, if p is a common prime factor of the order of $\mathfrak{S}$ and that of $\mathfrak{A}$, we consider any p-Sylow subgroup $\mathfrak{S}_p$ of $\mathfrak{G}$. Let $S\cdot A\neq e$ be an element of the center of $\mathfrak{S}_p$, S and A being elements of $\mathfrak{S}$ and $\mathfrak{A}$ respectively, and let P be any element of the center of a p-Sylow subgroup of $\mathfrak{S}$ which is contained in $\mathfrak{S}_p$. Then $P^{-1}\cdot S\cdot A\cdot P=S\cdot P^{-1}\cdot A\cdot P=S\cdot A$, whence $P^{-1}\cdot A\cdot P=A$. Therefore if $A\neq e$, the centralizer $\mathfrak{Z}(P)$ of P in $\mathfrak{G}$ contains $\mathfrak{S}$ properly. Since $\mathfrak{S}$ is maximal, $\mathfrak{Z}(P)=\mathfrak{G}$ and $\{P\}$ is an abelian normal subgroup of $\mathfrak{G}$. Then $\mathfrak{G}$ is solvable. If $A=e$, the centralizer $\mathfrak{Z}(S)$ of S in $\mathfrak{G}$ contains $\mathfrak{S}$ properly. Since $\mathfrak{S}$ is maximal, $\mathfrak{Z}(S)=\mathfrak{G}$ and $\{S\}$ is an abelian normal subgroup of $\mathfrak{G}$. Then $\mathfrak{G}$ is solvable.

Let p be a prime factor of the order of S. Then we can suppose that $\mathfrak{G}$ has no p-normality in the sense of O. GRÜN[3]). In fact, if $\mathfrak{G}$ is p-normal, then, by a theorem of O. GRÜN, the p-factor commutator subgroup of $\mathfrak{G}$ is isomorphic to that of the normalizer $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))$ of the center $\mathfrak{C}(\mathfrak{S}_p)$ of a p-Sylow subgroup of $\mathfrak{G}$. Since $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))\supset\mathfrak{S}$ and $\mathfrak{S}$ is maximal, $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))=\mathfrak{G}$ or $=\mathfrak{S}$. If $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))=\mathfrak{G}$, $\mathfrak{C}(\mathfrak{S}_p)$ is an abelian normal subgroup of $\mathfrak{G}$ and $\mathfrak{G}$ is

[1]) J. SZÉP, On factorisable, not simple groups, *these Acta*, **13** (1950), pp. 239—241.

[2]) J. SZÉP and L. RÉDEI, On factorisable groups, *these Acta*, **13** (1950), pp. 235—238.

[3]) H. ZASSENHAUS, *Lehrbuch der Gruppentheorie*. I. (Leipzig, 1937), p. 135.

solvable. If $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))=\mathfrak{S}$, then $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))\neq\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))'(p)$, where $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))'(p)$ is the p-commutator subgroup of $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))$. Therefore $\mathfrak{G}\neq\mathfrak{G}'(p)$, where $\mathfrak{G}'(p)$ is the p-commutator subgroup of $\mathfrak{G}$. Since $\mathfrak{G}'(p)\supset\mathfrak{A}$, $\mathfrak{G}'(p)=(\mathfrak{G}'(p)\cap\mathfrak{S})\cdot\mathfrak{A}$ and therefore $\mathfrak{G}'(p)$ solvable by induction hypothesis. Hence $\mathfrak{G}$ is solvable.

Last we can suppose that $\mathfrak{S}=\mathfrak{S}_p$ is a p-group. In fact, since $\mathfrak{G}$ is not p-normal, $\mathfrak{C}(\mathfrak{S}_p)$ is contained in at least two distinct p-Sylow subgroups of $\mathfrak{G}$ one of which may be $\mathfrak{S}_p$ itself. Therefore $\mathfrak{C}(\mathfrak{S}_p)$ is contained in at least two distinct conjugate subgroups of $\mathfrak{S}$ in $\mathfrak{G}$ one of which may be $\mathfrak{S}$ itself. Let $\mathfrak{S}^A$ be the other subgroup, where A is an element of $\mathfrak{A}$. Since $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))\supset\mathfrak{S}$ and $\mathfrak{S}$ is maximal, $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))=\mathfrak{G}$ or $=\mathfrak{S}$. If $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))=\mathfrak{G}$, then $\mathfrak{C}(\mathfrak{S}_p)$ is an abelian normal subgroup of $\mathfrak{G}$ and $\mathfrak{G}$ is solvable. If $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))=\mathfrak{S}$, then $\mathfrak{H}_p(\mathfrak{S})=\mathfrak{H}_p^A(\mathfrak{S})$, since $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))\supset\mathfrak{H}_p(\mathfrak{S})$ and $\mathfrak{N}(\mathfrak{C}(\mathfrak{S}_p))\supset\mathfrak{H}_p^A(\mathfrak{S})$, where $\mathfrak{H}_p(\mathfrak{S})$ is the p-Sylow complement of $\mathfrak{S}$, and since $\mathfrak{S}$ is nilpotent. Then the normalizer $\mathfrak{N}(\mathfrak{H}_p(\mathfrak{S}))$ of $\mathfrak{H}_p(\mathfrak{S})$ contains $\mathfrak{S}$ properly and coincides with $\mathfrak{G}$. Then $\mathfrak{H}_p(\mathfrak{S})$ is a nilpotent normal subgroup of $\mathfrak{G}$ and $\mathfrak{G}$ is solvable.

Then, by a theorem of W. BURNSIDE[4]), $\mathfrak{G}$ is not simple. Let $\mathfrak{N}$ be a proper normal subgroup of $\mathfrak{G}$ distinct from $\{e\}$. If $\mathfrak{S}\cdot\mathfrak{N}\neq\mathfrak{G}$, then $\mathfrak{S}\cdot\mathfrak{N}=$ $=\mathfrak{S}\cdot(\mathfrak{A}\cap\mathfrak{S}\cdot\mathfrak{N})$ and $\mathfrak{S}\cdot\mathfrak{N}$ is solvable by induction hypothesis. Therefore $\mathfrak{N}$ is a solvable normal subgroup of $\mathfrak{G}$ and $\mathfrak{G}$ is solvable. If $\mathfrak{S}\cdot\mathfrak{N}=\mathfrak{G}$, then the index of $\mathfrak{N}$ in $\mathfrak{G}$ is a power of p and $\mathfrak{N}$ contains $\mathfrak{A}$. Then $\mathfrak{N}=(\mathfrak{N}\cap\mathfrak{S})\cdot\mathfrak{A}$ and $\mathfrak{N}$ is solvable by induction hypothesis. Therefore $\mathfrak{N}$ is a solvable normal subgroup of $\mathfrak{G}$, whence $\mathfrak{G}$ is solvable, q. e. d.

(II) *Let $\mathfrak{G}$ be a factorizable group such that*

$$\mathfrak{G}=\mathfrak{S}\cdot\mathfrak{P}$$

where $\mathfrak{S}$ is nilpotent and $\mathfrak{P}$ is a p-group. Then $\mathfrak{G}$ is solvable.

Proof. The induction argument can be used with respect to the order of $\mathfrak{G}$; thus we may assume that every proper factor group of $\mathfrak{G}$ is solvable, and we have only to prove the existence of a solvable normal subgroup of $\mathfrak{G}$.

By a theorem of W. BURNSIDE, $\mathfrak{G}$ is not simple. Let $\mathfrak{N}$ be a proper normal subgroup of $\mathfrak{G}$ distinct from $\{e\}$. If $\mathfrak{P}\cdot\mathfrak{N}\neq\mathfrak{G}$, then $\mathfrak{P}\cdot\mathfrak{N}=(\mathfrak{P}\cdot\mathfrak{N}\cap\mathfrak{S})\cdot\mathfrak{P}$ and $\mathfrak{P}\cdot\mathfrak{N}$ is solvable by induction hypothesis. Therefore $\mathfrak{N}$ is a solvable normal subgroup of $\mathfrak{G}$ and $\mathfrak{G}$ is solvable. If $\mathfrak{P}\cdot\mathfrak{N}=\mathfrak{G}$, then the index of $\mathfrak{N}$ in $\mathfrak{G}$ is a power of p. Let $\mathfrak{H}_p(\mathfrak{S})$ be a p-Sylow complement of $\mathfrak{S}$. Then $\mathfrak{N}=\mathfrak{H}_p(\mathfrak{S})\cdot\mathfrak{S}_p(\mathfrak{N})$ where $\mathfrak{S}_p(\mathfrak{N})$ is a p-Sylow subgroup of $\mathfrak{N}$ and $\mathfrak{N}$ is solvable by our induction hypothesis. Therefore $\mathfrak{N}$ is a solvable normal subgroup of $\mathfrak{G}$ and $\mathfrak{G}$ is solvable. q. e. d.

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[4]) A. SPEISER, *Theorie der Gruppen von endlicher Ordnung* (Berlin, 1923), p. 136.