An individual ergodic theorem for non-commutative transformations.

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This note contains a proof of an individual ergodic theorem for a non-commutative family of measure preserving flows.*)

§ 1.

Theorem 1. *Let S be a set of finite Lebesgue measure, and $U_1^{\lambda_1}$, $U_2^{\lambda_2}, \dots U_k^{\lambda_k}$ a set of one-parameter measure preserving transformations of S onto itself. Let $f(x)$ be a real-valued function defined on S, measurable and such that the integral*

$$(1) \qquad \int_S f(x)\,\{\log^+ f(x)\}^{k-1}\,dx$$

is finite. Then the limit

$$(2) \qquad \lim_{\Lambda_1,\dots,\Lambda_k\to\infty} \frac{1}{\Lambda_1\dots\Lambda_k}\int_0^{\Lambda_1}\cdots\int_0^{\Lambda_k} f(U_1^{\lambda_1}U_2^{\lambda_2}\cdots U_k^{\lambda_k}x)\,d\lambda_1\cdots d\lambda_k$$

exists and is finite for almost every $x\in S$.

I omit the familiar conditions concerning the measurability of $f(U_1^{\lambda_1}\cdots U_k^{\lambda_k}x)$ in the product space of the λ's and of x. They guarantee, in particular, the existence of the integrals in (2) for almost all $x\in S$.

We may assume that $f\geqq 0$. By $\bar{f}(\xi)$ we shall denote the *decreasing real* rearrangement of $f(x)$. Thus, if $M(y)$ denotes the measure of the set of points where $f(x)\geqq y$, then $\bar{f}(\xi)$ is the inverse function of $M(y)$, e. g. normalized by the condition that $2\bar{f}(\xi)=\bar{f}(\xi+0)+\bar{f}(\xi-0)$ for all ξ. We need the following result of PITT(1) which we state as

*) See also N. DUNFORD, An individual ergodic theorem for non-commutative transformations, *these Acta*, **14** (1951), pp. 1–4.

1) H. R. PITT, Some generalizations of the ergodic theorem, *Proceedings Cambridge Philosophical Society*, **38** (1942), pp. 325–343, esp. p. 326.

Lemma 1. *Let T^λ be a one parameter group of measure-preserving transformations of S, and let $f(x)$ be a non-negative and integrable function of $x \in S$. Let*

$$F(x) = \sup_{\Lambda > 0} \frac{1}{\Lambda} \int_0^\Lambda f(T^\lambda x)\, d\lambda$$

Then

$$\bar{F}(y) \leqq \frac{1}{y} \int_0^y \bar{f}(\xi)\, d\xi \qquad (y > 0). \tag{3}$$

A few simple consequences of Lemma 1 will be stated here as

Lemma 2. *Under the assumptions of Lemma 1,*

$$\int_S F^p(x)\, dx \leqq A_p \int_S f^p(x)\, dx \qquad (p > 1), \tag{4}$$

$$\left(\int_S F^\alpha(x)\, dx\right)^{1/\alpha} \leqq A_\alpha \int_S f(x)\, dx \qquad (0 < \alpha < 1), \tag{5}$$

$$\int_S F(x) \{\log^+ F(x)\}^\beta\, dx \leqq 2 \int_S f(x) \{\log^+ f(x)\}^{\beta+1}\, dx + A_\beta \qquad (\beta = 0, 1, 2, \dots). \tag{6}$$

All the constants A here and hereafter (not necessarily the same at every occurrence) depend only on the variables displayed in the subscripts and (in some cases) on the number $a =$ measure of S. Inequality (4) will not be needed in the sequel and is stated here merely to give a perspective to the remaining inequalities. It is a familiar consequence of (2), for

$$\left\{\int_S F^p(x)\, dx\right\}^{1/p} = \left\{\int_0^a \bar{F}^p(y)\, dy\right\}^{1/p} \leqq \left\{\int_0^a \left[\frac{1}{y}\int_0^y \bar{f}(\xi)\, d\xi\right]^p dy\right\}^{1/p} \leqq$$

$$\leqq A_p \left\{\int_0^a \bar{f}^p(y)\, dy\right\}^{1/p} = A_p \left\{\int_S f^p(x)\, dx\right\}^{1/p},$$

and the relation between the third and fourth members here is the very well known inequality of Hardy. Similarly we establish inequality (5)[2]). It will be needed for any fixed value of α, e. g. for $\alpha = \frac{1}{2}$, in which case the factor A_α can be written A.

To prove (6), let $\omega(u)$ be any function of $u \geqq 0$, non-negative, non-decreasing and convex. Then

$$\begin{aligned} \int_S \omega[F(x)]\, dx &= \int_0^a \omega[\bar{F}(x)]\, dx \leqq \int_0^a \omega\left[\frac{1}{y}\int_0^y \bar{f}(\xi)\, d\xi\right] dy \leqq \\ &\leqq \int_0^a \frac{dy}{y} \int_0^y \omega[\bar{f}(\xi)]\, d\xi = \int_0^a \omega[\bar{f}(\xi)]\, d\xi \int_\xi^a \frac{dy}{y} = \int_0^a \omega[\bar{f}(\xi)] \log(a/\xi)\, d\xi, \end{aligned} \tag{7}$$

[2]) See e. g. A. Zygmund, *Trigonometrical series* (Warszawa-Lwów, 1935), p. 245.

the inequality between the third and fourth member being that of JENSEN For a given $\xi > 0$ we now distinguish two possibilities

$$1^0 \quad \bar{f}(\xi) \geqq (a/\xi)^{1/2}, \quad 2^0 \quad \bar{f}(\xi) < (a/\xi)^{1/2}.$$

In case 1^0, $a/\xi \leqq \bar{f}^2(\xi)$; in particular $\bar{f}(\xi) \geqq 1$. Hence

$$\omega[\bar{f}(\xi)] \log (a/\xi) \leqq 2\omega [\bar{f}(\xi)] \log \bar{f}(\xi) = 2\omega [\bar{f}(\xi)] \log^+ \bar{f}(\xi).$$

It follows that the last integral in (7) does not exceed

$$(8) \qquad 2\int_0^a \omega[\bar{f}(\xi)] \log^+ |f(\xi)| \, d\xi + \int_0^a \omega[(a/\xi)^{1/2}] \log (a/\xi) \, d\xi.$$

The function $\omega(u) = u(\log^+ u)^\beta$ is non-decreasing for $u \geqq 0$ if $\beta \geqq 0$, and is also convex if $\beta = 0$ or $\beta \geqq 1$. Moreover, the second integral in (8) is then finite. This proves (6). (Remark. If only $\beta > 0$, the function $u(\log^+ u)^\beta$ is convex for $u \geqq e$, and a minor modification of the proof gives (6) for $\beta > 0$, provided the factor 2 on the right is replaced by A_β. We can even assume that $\beta > -1$, if we replace the function $\log^+ F$ by $\log(2+F)$. The cases $\beta = 0, 1, 2, \cdots$ are, however the only ones we shall need for the proof of the theorem).

Lemma 3. *Let $f(x) \geqq 0$ satisfy the assumptions of the Theorem, and let*

$$(8') \qquad F^*(x) = \sup_{\Lambda_1, \ldots \Lambda_k > 0} \frac{1}{\Lambda_1 \cdots \Lambda_k} \int_0^{\Lambda_1} \cdots \int_0^{\Lambda_k} f(U_1^{\lambda_1} \cdots U_k^{\lambda_k} x) \, d\lambda_1 \cdots d\lambda_k.$$

Then

$$(9) \qquad \left\{ \int_S [F^*(x)]^\alpha dx \right\}^{1/\alpha} \leqq A_{\alpha,k} \int_S f(x) \left\{ \log^+ f(x) \right\}^{k-1} dx + A_{\alpha,k} \quad (0 < \alpha < 1).$$

For let us set

$$F_1(x) = \sup_{\Lambda_1 > 0} \frac{1}{\Lambda_1} \int_0^{\Lambda_1} f(U_1^{\lambda_1} x) \, d\lambda_1,$$

$$F_2(x) = \sup_{\Lambda_2 > 0} \frac{1}{\Lambda_2} \int_0^{\Lambda_2} F_1(U_2^{\lambda_2} x) \, d\lambda_2,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F_k(x) = \sup_{\Lambda_k > 0} \frac{1}{\Lambda_k} \int_0^{\Lambda_k} F_{k-1}(U_k^{\lambda_k} x) \, d\lambda_k.$$

We note that

$$\frac{1}{\Lambda_1\cdots\Lambda_k}\int_0^{\Lambda_1}\cdots\int_0^{\Lambda_k} f(U_1^{\lambda_1}\cdots U_k^{\lambda_k}x)\,d\lambda_1\cdots d\lambda_k \leqq$$

$$\leqq \frac{1}{\Lambda_2\cdots\Lambda_k}\int_0^{\Lambda_2}\cdots\int_0^{\Lambda_k} F_1(U_2^{\lambda_2}\cdots U_k^{\lambda_k})\,d\lambda_2\cdots d\lambda_k \leqq$$

$$\leqq \frac{1}{\Lambda_3\cdots\Lambda_k}\int_0^{\Lambda_3}\cdots\int_0^{\Lambda_k} F_2(U_3^{\lambda_3}\cdots U_k^{\lambda_k})\,d\lambda_3\cdots d\lambda_k \leqq \cdots \leqq$$

$$\leqq \frac{1}{\Lambda_k}\int_0^{\Lambda_k} F_{k-1}(U_k^{\lambda_k}x)\,d\lambda_k \leqq F_k(x).$$

Thus it is enough to prove (9) with F^* replaced by F_k. The new inequality, however, follows from (5) and (6). For

$$\Big(\int_S F_k^\alpha\,dx\Big)^{1/\alpha} \leqq A_\alpha\int_S F_{k-1}\,dx \leqq A_\alpha\int_S F_{k-2}\log^+ F_{k-2}\,dx + A_\alpha \leqq$$

$$\leqq A_\alpha\int_S F_{k-3}(\log^+ F_{k-3})^2\,dx + A_\alpha \leqq \cdots \leqq A_{\alpha,k}\int_S F_1(\log^+ F_1)^{k-2}\,dx + A_{\alpha,k} \leqq$$

$$\leqq A_{\alpha,k}\int_S f(\log^+ f)^{k-1}\,dx + A_{\alpha,k}.$$

The inequality (9) shows that under the assumption of the Theorem, the function F^* is finite almost everywhere. We shall deduce from it the existence of the limit (2) almost everywhere.

First of all we observe that it is enough to prove the existence of the limit (2) for f bounded (and non-negative). For let us replace in (9) α by $\frac{1}{2}$, and f by Mf, where M is a positive constant, and let us temporarily denote the second $A_{\alpha,k} = A_k$ in (9) by A_k'. Then dividing by M we get

$$(10)\qquad \Big(\int_S F^{*\frac{1}{2}}\,dx\Big)^2 \leqq A_k\int_S f\cdot(\log^+ Mf)^{k-1}\,dx + A_k'/M.$$

Let us select and fix M so large that $A_k'/M < \varepsilon/2$, and suppose that f is such that the first term on the right of (10) is also $< \frac{1}{2}\varepsilon$. Then $\int F^{*\frac{1}{2}}\,dx < \varepsilon^{1/2}$, and so the set of points x for which $F^* \geqq \varepsilon^{1/2}$ is of measure $< \varepsilon^{1/4}$. Suppose now that the existence of the limit (2) is established, almost everywhere, for any bounded f. Let us take a number $N > 0$, and let us make the decomposition $f = f_1 + f_2$, where $f_1(x) = \mathrm{Min}\,\{N, f(x)\}$. The finiteness of the integral (1) implies that of $\int_S f\cdot\{\log^+(fM)\}^{k-1}dx$, and so, if N is fixed sufficiently large, we shall have

$$A_k\int_S f_2\cdot\{\log^+(f_2M)\}^{k-1}\,dx < \frac{1}{2}\,\varepsilon.$$

The limit (2) exists, by assumption, almost everywhere if f there is replaced by f_1. It f is replaced by f_2, the upper bound of the resulting expression can exceed $\varepsilon^{1/4}$ only in a set of measure $< \varepsilon^{1/4}$. It follows that the limit (2) for $f = f_1 + f_2$ exists almost everywhere.

Thus the problem is reduced to proving the existence of the limit (2) for f bounded, say $f \leqq 1$, and here again we borrow an idea from PITT[8]. The proof in the case $k=2$ is already perfectly typical, and we may confine attention to this case and write U^λ, V^μ for $U_1^{\lambda_1}$, $U_2^{\lambda_2}$. We know (we take this result for granted) that

$$g(x) = \lim_{\Lambda \to \infty} \frac{1}{\Lambda} \int_0^\Lambda f(U^\lambda x)\, d\lambda$$

exists for almost every $x \in S$. Hence, given any $\varepsilon > 0$, we can find a set E, $|E| < \varepsilon$, such that

$$\left| \frac{1}{\Lambda} \int_0^\Lambda f(U^\lambda x)\, d\lambda - g(x) \right| < \varepsilon \qquad \text{for} \qquad x \in S - E, \Lambda > \Lambda_0(\varepsilon).$$

Let us replace here x by $V^\mu x$. Then

$$\left| \frac{1}{\Lambda} \int_0^\Lambda f(U^\lambda V^\mu x)\, d\lambda - g(V^\mu x) \right| < \varepsilon, \quad \text{if } V^\mu x \in S - E,$$

$$\left| \frac{1}{\Lambda} \int_0^\Lambda f(U^\lambda V^\mu x)\, dx - g(V^\mu x) \right| < 2, \quad \text{if } V^\mu x \in E.$$

Hence, if we denote by $h(x)$ the characteristic function of E, we get

$$\left| \frac{1}{\Lambda M} \int_0^\Lambda \int_0^M f(U^\lambda V^\mu x)\, d\lambda\, d\mu - \frac{1}{M} \int_0^M g(V^\mu x)\, d\mu \right| \leqq \varepsilon + \frac{2}{M} \int_0^M h(V^\mu x)\, d\mu.$$

Let

$$H(x) = \sup M^{-1} \int_0^M h(V^\mu x)\, d\mu.$$

The inequality

$$\bar{H}(y) \leqq y^{-1} \int_0^y \bar{h}(\xi)\, d\xi$$

shows that the set of y's for which $\bar{H}(y) \geqq \varepsilon^{1/2}$, and so also the set of points x for which $H(x) \geqq \varepsilon^{1/2}$, is of measure $\leqq \varepsilon^{1/2}$. It follows that for $\Lambda \geqq \Lambda_0$ and for all M we have the inequality

[8]) l. c. [1])

$$\left|\frac{1}{\Lambda M}\int_0^\Lambda\int_0^M f(U^\lambda V^\mu x)\,d\lambda\,d\mu - \frac{1}{M}\int_0^M g(V^\mu x)\,d\mu\right| \leqq \varepsilon + 2\sqrt{\varepsilon}$$

outside a set of measure $\leqq \varepsilon^{1/k}$, and independent of Λ, M.

Let us now observe that $M^{-1}\int_0^M g(V^\mu x)\,d\mu$ tends to a finite limit $g_1(x)$ almost everywhere, and so

$$\left|\frac{1}{M}\int_0^M g(V^\mu x)\,d\mu - g_1(x)\right| < \varepsilon$$

for x outside a set of measure $\leqq \varepsilon$ and independent of $M \geqq M_0(\varepsilon)$. Combining this with the previous inequality we finally obtain that

$$\left|\frac{1}{\Lambda M}\int_0^\Lambda\int_0^M f(U^\lambda V^\mu x)\,d\lambda\,d\mu - g_1(x)\right| < 2\varepsilon + 2\sqrt{\varepsilon}$$

for $\Lambda \geqq \Lambda_0, M \geqq M_0$, and outside a set independent of Λ, M and of measure $\leqq \varepsilon + \varepsilon^{1/k}$. This completes the proof of the Theorem.

Remarks. It is clear that if $f \varepsilon L^p, p > 1$, the function F^* of Lemma 3 also belongs to L^p, and $\int_S F^p(x)\,dx \leqq A_{p,k}\int_S f^p\,dx$. If $f(x)\{\log^+ f(x)\}^k$ is integrable, so is F^*, and

$$\int_S F(x)\,dx \leqq A_k \int_S f(x)\{\log^+ f(x)\}^k\,dx + A_k.$$

If $U_1^{\lambda_1}, \ldots, U_k^{\lambda_k}$ are commutative, we can complete Theorem 1 as follows.

Theorem 2. *Let S and $U_1^{\lambda_1}, \ldots, U_k^{\lambda_k}$ be the same as in Theorem 1. Let $L_1(t), \ldots, L_k(t)$ be positive functions defined for $t > 0$, non-decreasing, tending to 0 and $+\infty$ with t. Let l be a number not less than 1 and suppose that $\Lambda_1, \ldots, \Lambda_k$ satisfy the conditions*

(11) $$l^{-1}L_1(t) \leqq \Lambda_1 \leqq lL_1(t), \ldots, l^{-1}L_k(t) \leqq \Lambda_k \leqq lL_k(t).$$

Then for any function $f(x)$ integrable over S the limit (2) exists and is finite for almost every x.

It is enough to prove the following

Lemma 4. *Under the assumptions of Theorem 2, the function $F^*(x)$ defined in (8′) under conditions (11) and for $f \geqq 0$ satisfies the inequality*

(12) $$\left\{\int_S F^{*\alpha}(x)\,dx\right\}^{1/\alpha} \leqq A_{k,l,\alpha}\int_S f(x)\,dx \qquad (0 < \alpha < 1).$$

For then we make the usual decomposition $f = f_1 + f_2$, where f_1 is bounded and $\int_S f_2\,dx$ small.

Lemma 4 will follow if we prove the following result analogous to Lemma 1 (compare also inequality (5)).

Lemma 5. *Let $f \geqq 0$. Under the assumptions of Theorem 2 and under conditions* (11), *we have*

$$\bar{F}^*(y) \leqq (A_k/y) \int_0^y \bar{f}(\xi)\, d\xi \qquad (y > 0). \tag{13}$$

Without entering into a detailed proof of the lemma, one may stress the following points. The familiar proofs, like WIENER's or PITT's, of the ergodic theorem of BIRKHOFF are based on covering lemmas, like VITALI's or SIERPINSKI's. In our case we need the following covering lemma, in which for simplicity we consider sets of points (λ, μ) in the plane.

Lemma 6. *Let $h(t)$ and $k(t)$ be two positive functions defined for $t > 0$, non-decreasing and tending to 0 and $+\infty$ with t. Let E be a plane set whose outer measure $|E|$ is finite and positive. Suppose that to every point $(\lambda, \mu) \in E$ corresponds a rectangle $R = R_{\lambda,\mu}$ with lower left corner at (λ, μ), with sides parallel to the axes, and of lengths contained respectively between $l^{-1}h(t)$ and $lh(t)$, and between $l^{-1}k(t)$ and $lk(t)$, where $t = t(\lambda, \mu)$ varies with the point. Then there is a finite number of rectangles R_{λ_1,μ_1}, $R_{\lambda_2,\mu_2}, \dots, R_{\lambda_n,\mu_n}$ such that*

$$\sum_{j=1}^{n} |R_{\lambda_j,\mu_j}| \geqq A\,|E|. \tag{14}$$

For $l = 1$, the proof is known[4]) (there only rectangles with center at (λ, μ) are considered, but the proof remains unaffected). The constant A_l is then an absolute constant A. For $l > 1$ the result then follows immediately by considering rectangles $R'_{\lambda,\mu}$ with lower left corner at (λ, μ) and with sides $l^{-1}h(t)$ and $l^{-1}k(t)$, since obviously $|R| > |R'|$.

If we assume that for every $(\lambda, \mu) \in E$ there are rectangles R with t arbitrarily small, then applying the lemma a large (but finite) number of times we may cover E, except for a subset of arbitrarily small measure with rectangles R of the family. From this it follows without difficulty that the integral of any $f(\lambda, \mu) \in L$ is at almost every point differentiable with respect to rectangles R.[5])

From Lemma 6, one easily obtains the following result which is an analogue of PITT's Lemma 2[6]).

Lemma 7. *Let $\Lambda_0 > 0$, and let $g(\lambda_1, \lambda_2)$ be non-negative and integrable over any finite portion of the quadrant $\lambda_1 \geqq 0, \lambda_2 \geqq 0$. Let Λ_1, Λ_2 be the func-*

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[4]) See A. ZYGMUND, On the summability of multiple Fourier series, *American Journal of Math.*, **69** (1947), p. 838.

[5]) The fact is not new; it is explicitly stated in B. JESSEN, J. MARCINKIEWICZ and A. ZYGMUND, Note on the differentiability of multiple integrals, *Fundamenta Math.*, **25** (1935), pp. 217—234.

[6]) l. c. [1]), p. 327.

tions of Theorem 2 (for $k=2$) and suppose that for almost all (μ_1, μ_2)

$$\sup_{0<\Lambda_1, \Lambda_2 \leqq \Lambda_0} \frac{1}{\Lambda_1 \Lambda_2} \int_0^{\Lambda_1} \int_0^{\Lambda_2} g(\mu_1+\lambda_1, \mu_2+\lambda_2)\, d\lambda_1\, d\lambda_2 \geqq \alpha.$$

Then

$$\liminf_{\Lambda_1, \Lambda_2 \longrightarrow +\infty} \frac{1}{\Lambda_1 \Lambda_2} \int_0^{\Lambda_1} \int_0^{\Lambda_2} g(\lambda_1, \lambda_2)\, d\lambda_1\, d\lambda_2 \geqq A\alpha.$$

Lemma 5 follows from Lemma 7 exactly as in the case of one variable[7]), if one uses Lemma 6 and the differentiability theorem mentioned on the preceding page.

There is an intermediate result between Theorems 1 and 2, the analogue of which for differentiability of integrals is given in JESSEN, MARCINKIEWICZ and ZYGMUND[8]). The assumption is that $f(x)$ is measurable, and the integral

$$\int_S |f(x)|\, \{\log^+ |f(x)|\}^r\, dx$$

finite, where r is an integer satisfying the inequality

$$0 \leqq r \leqq k-1.$$

In that case, the limit (2) still exists almost everywhere, provided $k-r$ of the Λ_j satisfy conditions (11) while the remaining Λ's tend to $+\infty$ independently of one another. There is no need to give the details of the proof.

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[7]) See PITT, l. c., pp. 327–328.

[8]) l. c. [5]).