# A remark on the Jacobson radical.

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In this note we shall give a group-theoretic characterization of the Jacobson radical[1]) of a ring with identity.

**1.** We shall need an obvious generalization of the $\Phi$-subgroup[2]) of a group to groups with operator domains.

Let $G$ be a non-trivial group, commutative or not, and let $G$ have a (right) operator domain $\Omega$. The $\Omega$-subgroup generated by the elements $a, b, \ldots$ of $G$ will be denoted by $\{a, b, \ldots\}_\Omega$. If $\{a, b, \ldots\}_\Omega = G$, we say that $a, b, \ldots$ form an $\Omega$-generator-system of $G$. Just as in the case of groups without operators, it is easy to see that the set $\Phi(\Omega)$ of all the elements of $G$ which may be omitted from each $\Omega$-generator-system of $G$ is a subgroup. Moreover, $\Phi(\Omega)$ is an $\Omega$-subgroup, for $x \in \Phi(\Omega)$ and $\alpha \in \Omega$ imply $x\alpha \in \Phi(\Omega)$. In fact, if $\{x\alpha, K\}_\Omega = G$ for a certain subset $K$ of $G$, then $\{x, K\}_\Omega = G$ and therefore $\{K\}_\Omega = G$, i. e. $x\alpha \in \Phi(\Omega)$. This uniquely determined subgroup $\Phi(\Omega)$ of $G$ will be called the $\Phi(\Omega)$-subgroup of $G$. One may easily conclude that $\Phi(\Omega)$ is the intersection of the whole group $G$ with all maximal $\Omega$-subgroups of $G$.

**2.** Now let $R$ be a ring with an identity 1 and consider the additive group $R^+$ of $R$ as an operator-group with the right operator domain $R = \Omega$. The $R$-subgroups of $R^+$ are just the right ideals of the ring $R$.

Recall that the Jacobson radical of a ring $R$ is defined as the union of all right ideals of $R$ containing only right quasi-regular elements, and a right quasi-regular element $a$ may be defined by the property of having the form $x + ax$ for some $x \in R$, or, in rings with identity, of being contained in the right ideal[3]) $(1+a)_r$.

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[1]) The Jacobson radical of a ring was introduced in N. JACOBSON, The radical and semi-simplicity for arbitrary rings, *American Journal of Math.*, 67 (1945), pp. 300–320.

[2]) For the $\Phi$-subgroup of a group see: H. ZASSENHAUS, *The theory of groups* (New York, 1949), pp. 47–48.

[3]) If $K$ is a subset of $R$, the right ideal generated by $K$ will be denoted by $(K)_r$. The sign $\subset$ will denote inclusion, not necessarily a proper one.

We are going to prove the following

Theorem[4]). *The $\Phi(R)$-subgroup of $R^+$ is equal to the Jacobson radical $J$ of the ring $R$.*

*First proof.* N. Jacobson has proved[5]) that in rings with identity the Jacobson radical $J$ coincides with the intersection of all maximal right ideals of the ring $R$. Since the maximal right ideals of $R$ are obviously the maximal $R$-subgroups of $R^+$, by the last remark in 1 we are led to our assertion.

We shall give even another proof of our theorem, a proof which is based immediately on the definitions and does not make use of any previous result on the Jacobson radical.

*Second proof.* Let the right ideal $(a)_r$ contain an element $b$ which is not right quasi-regular and consider the right ideal $A=(a, 1+b)_r$. Since $bx \in (a)_r$ and $x+bx \in (1+b)_r$ for each $x \in R$, we obtain that $x=(x+bx)-bx$ belongs to $A$ and hence $A=R$. But $(1+b)_r \neq R$, that is, from the $R$-generator-system $a, 1+b$ the element $a$ can not be omitted, considering that $b$ does not belong to $(1+b)_r$, $b$ being not a right quasi-regular element. This proves that $\Phi(R) \subset J$.

Conversely, let all the elements of $(a)_r$ be right quasi-regular and $R=(a, K)_r=(a)_r+(K)_r$ for some subset $K$ of $R$. Since $R$ has an identity 1, we get $-b+r=1$ for some $b \in (a)_r$ and $r \in (K)_r$. Clearly, for this $b$ we have $R=(b, K)_r$. Now take into account that by hypothesis $b$ is a right quasi-regular element and besides that $(K)_r$ contains $r=1+b$, consequently, $b \in (1+b)_r \subset (K)_r$. Therefore it follows that $b$ may be deleted and hence $(K)_r=R$. This shows that $J \subset \Phi(R)$ and hence the proof of the theorem is completed.

Finally let us remark that if we omit the hypothesis of the existence of 1 in $R$, the theorem in general fails to hold. For example, in the ring of all even rational integers modulo 4, consisting of the elements 0 and 2, the Jacobson radical is the whole ring, while the $\Phi(R)$-subgroup consists of the single element 0.

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[4]) Professor L. Rédei has kindly called my attention to the fact that this theorem also holds if the ring is assumed only to have a one-sided unit element. The proof remains the same.

[5]) See Jacobson, loc. cit. [1]), in particular p. 311.