On immediate inclusion in partially ordered sets and the construction of homology groups for metric lattices.

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1. Introduction. The notion of immediate inclusion ("covering") is defined in any partially ordered set and thus any partially ordered set gives rise to an immediate inclusion set. In this note, we obtain conditions on a binary relation which characterize the immediate inclusion relations arising from partial orderings. We next point out characterizations of modularity and distributivity in lattices in terms of immediate inclusion. In the last section, we discuss the possibility of generalizing the homology theory of complexes to apply to certain types of normed lattices where the "dimension function" is not necessarily integer-valued.

2. The immediate inclusion relation of a partial ordering. Let P be a set partially ordered by $\leqq$; that is,

PO1 $a \leqq a$,

PO2 $a \leqq b$ and $b \leqq a$ imply $a = b$,

PO3 $a \leqq b$ and $b \leqq c$ imply $a \leqq c$.

We define as usual: $a < b$ if $a \leqq b$ and $a \neq b$; $a \geqq b$ if $b \leqq a$; $a > b$ if $a \geqq b$ and $a \neq b$. We write $a \rightarrow b$, read "a is immediately included in b (or a immediately precedes b)", provided $a < b$ and $a \leqq c \leqq b$ implies $a = c$ or $b = c$.

Lemma 2.1. *If P is a partially ordered set and $\rightarrow$ is the immediate inclusion relation of P, then ($a \overset{\rightarrow}{=} b$ means $a \rightarrow b$ or $a = b$):*

II1 $a \overset{\rightarrow}{=} x_1 \overset{\rightarrow}{=} x_2 \overset{\rightarrow}{=} \cdots \overset{\rightarrow}{=} x_\beta \overset{\rightarrow}{=} a$ *implies $x_\alpha = a$ for all $\alpha \leqq \beta$, where β is any ordinal number;*

II2 $a \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots \rightarrow x_\beta \rightarrow b$ *implies $a \not\rightarrow b$ where β is a non-zero ordinal number.*

Proof. II1 follows from PO2, PO3, and transfinite induction since $x_\alpha \rightarrow x_\beta$ implies $x_\alpha \leqq x_\beta$. II2 is immediate from the definition of $\rightarrow$, PO3, and transfinite induction.

Lemma 2.2. *If P is a set and $\to$ is a binary relation on P satisfying* II1 *and* II2 *and if one defines $a<b$ if and only if there is a well-ordered terminating sequence $x_1, x_2, \ldots, x_\beta$ (β is any ordinal number, the sequence need not be countable) with $a \to x_1 \to x_2 \to \cdots \to x_\beta \to b$, then P is partially ordered by $\leqq$ and the immediate inclusion relation arising from $\leqq$ coincides with $\to$.*

Proof. PO1 follows from definition of $a \leqq b$ as $a<b$ or $a=b$. PO2 is obtained from II1 since $a \leqq b$ and $b \leqq a$ imply a chain of the form $a \rightleftarrows x_1 \rightleftarrows x_2 \rightleftarrows \cdots \rightleftarrows x_\beta \rightleftarrows b \rightleftarrows y_1 \rightleftarrows y_2 \rightleftarrows \cdots \rightleftarrows y_\gamma \rightleftarrows a$. PO3 is immediate from the definition of $\leqq$. The coincidence of $\to$ and the immediate inclusion relation of $\leqq$ is now obtained immediately from II2.

Theorem 2.1. *A binary relation defined on a set is the immediate inclusion relation of a partial ordering of the set if and only if conditions* II1 *and* II2 *are satisfied.*

It is clear that an isomorphism between two partially ordered sets is necessarily an isomorphism for their immediate inclusion relations. One enquires as to the existence of partially ordered sets for which every II isomorphism is a PO isomorphism. A partial answer is given in

Theorem 2.2. *If P and P' are partially ordered sets in which every ordered pair of elements are joined by a well-ordered principal chain (a chain in which no further interpolation is possible) then every* II *isomorphism of P and P' is a* PO *isomorphism.*

3. Immediacy characterizations of modularity and distributivity in lattices. The possession of meets and joins in a partially ordered set is not closely governed by the immediacy of precession. One may, of course, have lattices in which no pair of elements are immediately ordered. It is somewhat surprising then that the algebraic properties of modularity and distributivity may be characterized in terms of immediate inclusion. In this section, L denotes an arbitrary lattice.

Lemma 3.1. (Birkhoff [1], p. 66) *If L is modular then*

B $\quad$ *$a \wedge b \to a$ and $a \wedge b \to b$ imply $a \to a \vee b$ and $b \to a \vee b$, and dually.*

Since modularity is a hereditary property we have

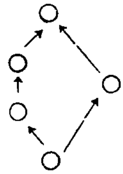
Lemma 3.2. *If L is modular then condition* B *subsists in every sublattice of L.*

One easily constructs examples to show that condition B itself is not hereditary. Thus, to obtain the desired characterization of modularity, we must require condition B for every sublattice.

Theorem 3.1. *A necessary and sufficient condition that L be modular is that condition* B *subsist for every sublattice of L.*

Proof. The necessity is given by Lemma 3. 2. To establish sufficiency we note that if L is not modular, it contains the non-modular picture

(1)

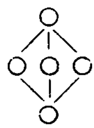


as a sublattice and condition B fails in this sublattice.

Lemma 3. 3. *If L is distributive and $a_1, a_2, \ldots, a_n$* (n a positive integer) *are distinct immediate predecessors of a then the meets of $a_1, a_2, \ldots, a_n$ taken k at a time are distinct; $k = 1, 2, \ldots, n$. The dual proposition is also valid.*

Proof. The proposition is obvious for $n = 1, 2$. Its falsity for $n = 3$ implies the non-distributive picture

(2)



as a sublattice. The proof is completed by induction, for supposing the proposition valid for $n \leqq k$ $(k \geqq 3)$ but false for $n = k + 1$ leads to a non-modular sublattice. The dual proposition follows from the self-duality of distributive lattices.

For use in the next section we write down the

Corollary. *If L is distributive and an element has n distinct immediate predecessors (immediate successors) then L contains a descending chain (an ascending chain) of at least n distinct elements.*

We shall refer to the property of a lattice expressed in Lemma 3. 3 as condition D.

Theorem 3. 2. *A modular lattice L is distributive if and only if condition* D *subsists in every sublattice.*

Proof. Since distributivity is hereditary, Lemma 3. 3 shows the "only if" part. Suppose L is not distributive. Then L contains the non-distributive picture (2) as a sublattice and condition D fails in this sublattice.

If we strengthen condition D to

D′ *If $a_1, a_2, \ldots, a_n$ are distinct immediate predecessors of a, then condition* D *holds and the chain $a, a_1, a_1 \wedge a_2, a_1 \wedge a_2 \wedge a_3, \ldots, \bigwedge_{i=1}^{n} a_i$ is principal; and dually,*

we may obtain from Theorems 3. 1 and 3. 2 a characterization of distributivity without the presumption of modularity.

Theorem 3. 3. *L is distributive if and only if condition* D′ *subsists in every sublattice.*

4. Homology groups of normed lattices. The notion of "is a face of" in the theory of complexes may be thought of as immediate inclusion in the set-theoretic ordering of the complex. This suggests the possibility of defining the notion of homology group for suitably restricted normed lattices in much te same manner as for complexes. We restrict attention to distributive normed lattices and discrete coefficient groups (although the use of topological coefficient groups appears to offer no additional difficulty). A discussion of homology groups for complexes may be found in (3) and the notion of a normed lattice as used here may be found in (2).

Since we wish to consider general chains, it is necessary that L be star-finite. We interpret "is a face of" as immediate inclusion and thus the Corollary to Lemma 3. 3 assures that L will be star-finite if we impose the finite ascending chain condition on L. The imposition of the finite descending chain condition on L would, of course, make L closure-finite. In the remainder of the paper, L denotes a distributive normed lattice with the finite ascending chain condition. This condition also assures that L has last element corresponding to the entire complex. We define $L^p = \{x \in L \,\|\, |x| = p\}$ and suppose that each L^p has a fixed well-ordering of ordinal type β_p so that L^p may be written $L^p = \{x^p_\alpha\}$; $\alpha < \beta_p$. Let G be an additive Abelian group. We shall employ the tensor convention for the formal summation on a repeated greek index placed once covariantly and once contravariantly. G^p is the set of all formal sums $g^\alpha x^p_\alpha, g^\alpha \in G$. G^p forms, of course, an Abelian group as the direct product of β_p replicas of G and G^p is called the group of p-chains of L over G. We define $\left\{\begin{matrix} q & \beta \\ p & \alpha \end{matrix}\right\}$ to be 0 unless $x^q_\beta \to x^p_\alpha$ or $x^p_\alpha \to x^q_\beta$ in which case the symbol must be 1 or -1. These symbols may be called the incidence numbers. One requires $\left\{\begin{matrix} q & \beta \\ p & \alpha \end{matrix}\right\} = \left\{\begin{matrix} p & \alpha \\ q & \beta \end{matrix}\right\}$. One next defines boundary operators. For $q < p$ let $C^p \in G^p$, $C^p = g^\alpha x^p_\alpha$ and define

$$\partial_q C^p = \left\{\begin{matrix} q & \beta \\ p & \alpha \end{matrix}\right\} g^\alpha x^q_\beta.$$

One sees that ∂_q is a homomorphism of G^p into G^q. The image of G^p is J^q_p, the group of p-bounding q-chains of L over G. The kernel of this homomorphism is written Z^p_q and is the group of p-cycles for G^q.

We would now be in a position to define homology groups except that it may happen that J^q_p is not necessarily a subgroup of Z^q_r for $r < q < p$! Let us examine this possibility.

Remark. For a given α and γ and $r<q<p$ there are at most two $x_\beta^q \in G^q$ with $\begin{Bmatrix} q & \beta \\ p & \alpha \end{Bmatrix} \begin{Bmatrix} r & \gamma \\ q & \beta \end{Bmatrix} \neq 0$ (β not summed). Otherwise, we would have a non-distributive sublattice of L. The only cases for discussion then are those where there is exactly one or are exactly two such x_β^q.

Remark. If there are exactly two x_β^q as in the preceding remark, then $q-r=p-q$. This follows immediately from the norm condition $|a\vee b|+$ $+|a\wedge b|=|a|+|b|$. We assume then that L has the additional property: If $q-r=p-q$, $r<q<p$ there are either two or none of the x_β^q as in the above remarks. If $q-r\neq p-q$ we assume there are no such x_β^q. One may then show by repeated transfinite inductions that the incidence numbers may be so chosen that for a given α and γ and $r<q<p$,

$$\begin{Bmatrix} q & \beta \\ p & \alpha \end{Bmatrix} \begin{Bmatrix} r & \gamma \\ q & \beta \end{Bmatrix} = 0 \qquad (\beta \text{ summed}).$$

Assuming the requirement of the last remark to be in effect, we have $\partial_r \partial_q C^p = 0$ for $r<q<p$ so that every boundary in L^q is a cycle and $J_p^q \subset z_r^q$.

The homology group H_{pr}^q may then be defined for $r<q<p$ as the factor group Z_r^q/J_p^q. (As usual, if G were a topological group we would take closures (topological) before taking factor groups.)

One sees then that there is a natural generalization of the notion of homology group from a complex to a suitably restricted normed lattice in which "dimensions" are not necessarily integral. Of course, H_{pr}^q as well as J_p^q and Z_r^q are invariants (up to isomorphism) under isometric isomorphisms of the lattices generating them. The moot (and unanswered) question is: What is the most general class (algebraically defined) of isometric correspondences between lattices of the type under consideration under which the homology groups are invariant? This amounts, in the case of complexes, to the Poincaré question as to the algebraic relations subsisting between arbitrary polyhedral decompositions of homeomorphic spaces.

References.

[1] Garret Birkhoff, *Lattice Theory* (revised edition, New-York, 1948).

[2] L. M. Blumenthal and D. O. Ellis, Notes on lattices, *Duke Math. Journal*, **16** (1949), pp, 585—590.

[3] S. Lefschetz, *Abstract complexes*, Lectures in Topology (University of Michigan Press, 1941), pp. 1–28.

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