On a theorem of L. Rédei and J. Szép concerning p-groups.

By NOBORU ITÔ in Nagoya (Japan).

In their recent paper[1]) L. RÉDEI and J. SZÉP obtained the following interesting result on p-groups: Let G be a p-group such that $G = \{H, A\}$, where H is a subgroup of G and A is an element of G. If $D(G)$[2]) contains $D(\{H, A^{p^2}\})$ properly then $D(G)$ also contains $D(\{H, A^p\})$ properly. Further they have made the following two conjectures: (1) The index of $(\{H, A^p\})$ in G is greater than that of $\{H, A^{p^2}\}$ in $\{H, A^p\}$. (2) The index of $D(\{H, A^p\})$ in $D(G)$ is greater than that of $D(\{H, A^{p^2}\})$ in $D(\{H, A^p\})$.

Now, in this note, we want to generalize a little more the above theorem of L. RÉDEI and J. SZÉP (§ 1) and settle negatively the above conjectures of them (§ 2).

§ 1.

Theorem. *Let G be a p-group and let F and H be the Frattini subgroup and any subgroup of G, respectively. If $D(G)$ contains $D(H)$ properly, then $D(G)$ also contains $D(FH)$ properly.*

Proof. We prove this assertion by an induction argument with respect to the index of H in G and the order of G.

First we can assume that H contains $D(G)$. In fact, if H does not contain $D(G)$, $HD(G)$ contains H properly, therefore, $HD(G)$ contains a subgroup K, in which H has the index p. Since H is normal in K, H contains $D(K)$. Further since the index of K in G is smaller than that of H in G and $D(G)$ contains $D(K)$ properly, $D(G)$ contains $D(FK)$ properly by the induction hypothesis. Since $D(KF)$ contains $D(HF)$, $D(G)$ contains $D(HF)$ properly. Therefore we can assume that H contains $D(G)$, that is, we can assume that H is normal in G.

Secondly we can assume that $D(H)$ is equal to E. In fact, if $D(H)$ is different from E, let us consider the factor group $G/D(H)$. Since the Frattini

[1]) L. RÉDEI—J. SZÉP, Über die endlichen nilpotenten Gruppen, *Monatshefte für Math.*, 55 (1951), pp. 200—205.

[2]) We denote by $D(X)$ the commutator subgroup of the group X.

subgroup $F(G/D(H))$ of $G/D(H)$ is equal to $F/D(H)$ and the order of $G/D(H)$ is smaller than that of G, $D(G)$ contains $D(FH)$ properly by the induction hypothesis. Therefore we can assume that $D(H)$ is equal to E, that is, H is abelian.

Thirdly we can assume that $D(G)$ is of order p. In fact, if $D(G)$ is not of order p, $D(G)$ contains properly a central subgroup C of order p and G/C is not abelian. Since the Frattini subgroup $F(G/C)$ of G/C is equal to F/C and the order of G/C is smaller than that of G, $D(G)$ contains $D(FH)$ properly by the induction hypothesis. Therefore we can assume that $D(G)$ is of order p. In particular the centre $Z(G)$ of G contains $D(G)$, that is, G is of class 2.

Finally, in a p-group G of class 2 such that $D(G)$ is of type $(p, p, \ldots, p)$, the subgroup $W(G)$, which is generated by all the p-th powers of the elements of G, is contained in $Z(G)$. In fact, $(A^p, B) = (A, B)^p = E$ for any elements A, B of G. Therefore in such a group $Z(G)$ contains $W(G)$, whence $Z(G)$ contains F, and FH is abelian. Thus we complete the proof of our assertion.

§ 2.

Now here are the counter-examples to the conjectures (1) and (2) of Rédei and Szép.

Example 1. Let G be a group of order p^{p^2+2} such that $G=\{A, B_1, B_2, \ldots, B_{p^2}\}$, where $A^{p^2} = B_1^p = B_2^p = \cdots = B_{p^2}^p = E$ and $A^{-1}B_1A = B_2, \ldots, A^{-1}B_{p^2}A = B_1$. Put $H = \{B_1, B_2, \ldots, B_p\}$. Then $a)$ the index of $\{H, A^p\}$ in G is equal to p and that of H in $\{H, A^p\}$ is equal to p^{p^2-p+1}; $b)$ the index of $D(\{H, A^p\})$ in $D(G)$ is equal to p^{p-1} and the order of $D(\{H, A^p\})$ is equal to $p^{p(p-1)}$

Example 2.[3]) Let G be a group of order p^{2p+2} such that $G = \{A, B_1, \ldots, B_{2p-1}, B_{2p}\}$, where $A^{p^2} = B_1^p = \cdots = B_{2p-1}^p = B_{2p}^p = E$ and $A^{-1}B_1A = B_1, B_2, \ldots, A^{-1}B_{2p-1}A = B_{2p-1}B_{2p}$, $A^{-1}B_{2p}A = B_{2p}$. Put $H = \{B_1, \ldots, B_p\}$. Then $a)$ the index of $\{H, A^p\}$ in G is equal to p and that of H in $\{H, A^p\}$ is equal to p^{p+1}; $b)$ the index of $D(\{H, A^p\})$ in $D(G)$ is equal to p^{p-1} and the order of $D(\{H, A^p\})$ is equal to p^p.

Mathematical Institute,
Nagoya University.

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3) This example is due to Mr. M. Nagata.