A theorem on the normalcy of completely continuous operators.

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Let A be a completely continuous operator in a Hilbert space $\mathfrak{H}$, and let the eigenvalues of the operator A and of the non-negative selfadjoint operator A^*A be denoted by α_i and $\varkappa_i$ $(i=1, 2, \ldots)$ respectively, which are so arranged that

(1) $$|\alpha_1| \geqq |\alpha_2| \geqq \cdots, \quad \varkappa_1 \geqq \varkappa_2 \geqq \cdots.$$

Long ago I. SCHUR [1] proved the following inequalities

(2) $$\lambda_1 + \lambda_2 + \cdots + \lambda_\nu \leqq \varkappa_1 + \varkappa_2 + \cdots + \varkappa_\nu \qquad (\nu = 1, 2 \ldots),$$

where

(3) $$\lambda_i = |\alpha_i|^2 \quad (i = 1, 2, \ldots).$$

Recently H. WEYL [2] showed that the inequalities

(4) $$\varphi(\lambda_1) + \varphi(\lambda_2) + \cdots + \varphi(\lambda_\nu) \leqq \varphi(\varkappa_1) + \varphi(\varkappa_2) + \cdots + \varphi(\varkappa_\nu) \quad (\nu = 1, 2, \ldots)$$

hold for every function $\varphi(x)$ which is defined on $[0, \infty)$ and increasing, such that $\varphi(e^\xi)$ is a convex function of ξ. More recently G. PÓLYA [3] gave an elementary proof of Weyl's inequality. These facts led me to extend a theorem of HARDY-LITTLEWOOD-PÓLYA [4] (see theorem 1). By means of this extension it can be shown that if $\psi(x)$ is any strictly increasing convex function defined on $[0, \infty)$, then the condition

(N) $$\lim_{n \to \infty} \left(\sum_{i=1}^{n} \psi(\varkappa_i) - \sum_{i=1}^{n} \psi(\lambda_i) \right) = 0$$

implies the normalcy of the operator A (see theorem 2). Of course this class of functions $\psi(x)$ is much narrower than the class of functions $\varphi(x)$ figuring in Weyl's theorem. We remark that our theorem is not true for the class of functions $\varphi(x)$. A counter example will be given at the end of this paper.

We start with the

Definition. A sequence of numbers $\alpha_1', \alpha_2', \ldots$ is said to be quasi-majorised by a sequence of numbers $\alpha_1, \alpha_2, \ldots,$ if the following conditions are satisfied

(i) $\alpha'_i \geqq 0, \qquad \alpha_i \geqq 0 \qquad (i=1,2,\ldots),$

(ii) $\alpha'_1 \geqq \alpha'_2 \geqq \ldots, \qquad \alpha_1 \geqq \alpha_2 \geqq \ldots,$

(iii) $\alpha'_1+\alpha'_2+\ldots+\alpha'_\nu \leqq \alpha_1+\alpha_2+\ldots+\alpha_\nu \qquad (\nu=1,2,\ldots),$

(iv) $$\lim_{n\to\infty}\left[\sum_{i=1}^{n}\alpha_i-\sum_{i=1}^{n}\alpha'_i\right]=0.$$

Theorem I. *Let $\{\alpha'_i\}$ and $\{\alpha_i\}$ be two sequences of numbers, such that $\alpha'_i \geqq 0$, $\alpha_i \geqq 0$ $(i=1,2,\ldots)$ and*

$$\alpha'_1 \geqq \alpha'_2 \geqq \ldots, \quad \alpha_1 \geqq \alpha_2 \geqq \ldots .$$

The sequence $\{\alpha'_i\}$ is quasi-majorised by the sequence $\{\alpha_i\}$ if and only if there exist two sequences of positive integers

$$n_1 \leqq n_2 \leqq \ldots, \quad \nu_1 < \nu_2 < \ldots \quad \text{with} \quad n_i \leqq \nu_i \qquad (i=1,2,\ldots),$$

so that for every positive integer k we can find ν_k^2 non-negative numbers

(5) $$p^{(k)}_{\mu\nu} \geqq 0 \qquad (\mu,\nu=1,2,\ldots,\nu_k)$$

such that

$$\sum_{\nu=1}^{\nu_k} p^{(k)}_{\mu\nu}=1 \ (\mu=1,2,\ldots,\nu_k), \qquad \sum_{\mu=1}^{\nu_k} p^{(k)}_{\mu\nu}=1 \quad (\nu=1,2,\ldots,\nu_k)$$

and

(6) $$\alpha'_\mu=\sum_{\nu=1}^{\nu_k} p^{(k)}_{\mu\nu}\,\alpha_\nu \ (1 \leqq \mu \leqq \nu_k,\ \mu \neq n_k) \quad \alpha'_{n_k} \leqq \sum_{\nu=1}^{\nu_k} p^{(k)}_{n_k\nu}\,\alpha_\nu,$$

furthermore

(7) $$\lim_{k\to\infty}\left[\sum_{\nu=1}^{\nu_k} p_{n_k\nu}\,\alpha_\nu-\alpha'_{n_k}\right]=0.$$

Proof. That the condition is sufficient, is evident. To prove the necessity we need the following lemmas.

Lemma I. *Suppose $\alpha_l-\alpha'_l$ is the first negative difference among the differences $\alpha_i-\alpha'_i$ $(i=1,2,\ldots)$, and $\alpha_k-\alpha'_k$ the last positive difference which precedes $\alpha_l-\alpha'_l$, i. e.*

$$\alpha'_\nu \leqq \alpha_\nu \qquad (\nu=1,2,\ldots,k-1),$$
$$\alpha'_k<\alpha_k,\ \alpha'_{k+1}=\alpha_{k+1},\ldots,\ \alpha'_{l-1}=\alpha_{l-1},\ \alpha'_l>\alpha_l.$$

If we take

(8) $$\alpha_k=\varrho+\tau,\ \alpha_l=\varrho-\tau,$$

and define σ by

(9) $$\sigma=\max(|\alpha'_k-\varrho|,\ |\alpha'_l-\varrho|),$$

then $0 \leqq \sigma < \tau \leqq \varrho$, and the sequence of numbers $\{\alpha'_i\}$ is quasi-majorised by *the sequence of numbers* $\{\alpha_i''\}$ *which is defined as*

$$\text{(T)}\qquad \begin{cases} \alpha_k'' = \dfrac{\tau+\sigma}{2\tau}\alpha_k + \dfrac{\tau-\sigma}{2\tau}\alpha_l, \\[2mm] \alpha_l'' = \dfrac{\tau-\sigma}{2\tau}\alpha_k + \dfrac{\tau+\sigma}{2\tau}\alpha_l, \\[2mm] \alpha_\nu'' = \alpha_\nu \qquad (\nu \neq k,\ \nu \neq l). \end{cases}$$

Moreover, for the sequence $\{\alpha_i''\}$ *so defined, at least one of the equalities* $\alpha_k'' = \alpha_k',\ \alpha_l'' = \alpha_l'$ *is true.*

This has been proved by HARDY, LITTLEWOOD and PÓLYA [3, p. 47—48] when both sequences are supposed to be null after a finite number of terms. However their proof remains valid for the present lemma almost word for word.

Suppose $\alpha_1 - \alpha_1' > 0$. Then, if we apply the transformation T (see above lemma) to the sequence $\{\alpha_i\}$ successively, the first element α_1 must be affected after a finite number of times. For, if not, we shall have

$$\text{(10)}\qquad \sum_{i=1}^{n}\alpha_i - \sum_{i=1}^{n}\alpha_i' \geqq \alpha_1 - \alpha_1' \qquad (n = 1, 2, \ldots),$$

which contradicts to the hypothesis that $\lim\limits_{n\to\infty}\left(\sum\limits_{i=1}^{n}\alpha_i - \sum\limits_{i=1}^{n}\alpha_i'\right) = 0.$

Let m be the least number of times for the first element α_1 to be affected in the successive application of the transformations T. Then the sequence $\{\alpha_i^*\}$ arising from $\{\alpha_i\}$ by the m-times successive application of the transformations T, will enjoy the following properties:

1) $\{\alpha_i'\}$ is quasi-majorised by $\{\alpha_i^*\}$.

2) There exist n^2 non-negative numbers

$$p_{\mu\nu} \geqq 0 \qquad (\mu, \nu = 1, 2, \ldots, n)$$

with

$$\sum_{\nu=1}^{n} p_{\mu\nu} = 1 \quad (\mu = 1, 2, \ldots, n), \qquad \sum_{\mu=1}^{n} p_{\mu\nu} = 1 \quad (\nu = 1, 2, \ldots, n),$$

so that

$$\alpha_\mu^* = \sum_{\nu=1}^{n} p_{\mu\nu}\alpha_\nu \quad (\mu = 1, 2, \ldots, n), \quad \alpha_\mu^* = \alpha_\mu \qquad (\mu = n+1, n+2, \ldots).$$

3)
$$\begin{cases} \alpha_\mu' = \alpha_\mu^* \quad (1 \leqq \mu \leqq n,\ \mu \neq 1,\ \mu \neq n_0), \\ \alpha_1' \leqq \alpha_1^*, \quad \alpha_{n_0}' \leqq \alpha_{n_0}^*, \end{cases}$$

where n_0 is an integer, $1 < n_0 \leqq n$. Moreover, at least one of the equalities $\alpha_1' = \alpha_1^*,\ \alpha_{n_0}' = \alpha_{n_0}^*$ is true.

4)
$$\sum_{i=1}^{\nu}\alpha_i - \sum_{i=1}^{\nu}\alpha_i' \geqq \alpha_1 - \alpha_1' \qquad (1 \leqq \nu < n).$$

Thus we have proved the

Lemma 2. *Let the sequence* $\{\alpha_i'\}$ *be quasi-majorised by the sequence* $\{\alpha_i\}$. *If* $\alpha_1-\alpha_1'>0$ *and if* $\alpha_l-\alpha_l'$ *is the first which is negative among the differences* $\alpha_i-\alpha_i'$ $(i=1,2,\ldots)$, *then there exist* n^2 $(n\geqq l)$ *non-negative numbers* $p_{\mu\nu}\geqq 0$ $(\mu,\nu=1,2,\ldots,n)$ *with*

$$\sum_{\nu=1}^{n} p_{\mu\nu}=1 \quad (\mu=1,2,\ldots,n), \quad \sum_{\mu=1}^{n} p_{\mu\nu}=1 \quad (\nu=1,2,\ldots,n),$$

such that

$$\text{(11)} \qquad \begin{cases} \alpha_\mu' = \sum_{\nu=1}^{n} p_{\mu\nu}\alpha_\nu & (\mu\neq 1,\ \mu\neq n_0), \\ \alpha_1' \leqq \sum_{\nu=1}^{n} p_{1\nu}\alpha_\nu, & \\ \alpha_{n_0}' \leqq \sum_{\nu=1}^{n} p_{n_0\nu}\alpha_\nu, & \end{cases}$$

where n_0 *is a certain positive integer,* $1<n_0\leqq n$, *and*

$$\text{(12)} \qquad \sum_{i=1}^{\nu}\alpha_i-\sum_{i=1}^{\nu}\alpha_i'\geqq \alpha_1-\alpha_1' \qquad (1\leqq \nu<n).$$

Moreover, among the two inequalities in (11), *at least one equality sign holds.*

We now proceed to the proof of the necessity part of theorem I.

By means of lemma 2, the sequence of transformation matrices

$$\text{(13)} \qquad (p_{\mu\nu}^{(k)})_{\mu,\nu=1,2,\ldots,\nu_k} \qquad (k=1,2,\ldots)$$

required in the theorem, is obtained subsequently. It remains to show

$$\text{(14)} \qquad \lim_{k=\infty}\left(\sum_{\nu=1}^{\nu_k} p_{n_k\nu}^{(k)}\alpha_\nu-\alpha_{n_k}'\right)=0.$$

To prove this, we suppose to the contrary that

$$\overline{\lim_{k\to\infty}}\left(\sum_{\nu=1}^{\nu_k} p_{n_k\nu}^{(k)}\alpha_\nu-\alpha_{n_k}'\right)=d>0,$$

i. e.

$$\sum_{\nu=1}^{\nu_k} p_{n_k\nu}^{(k)}\alpha_\nu-\alpha_{n_k}'\geqq \frac{d}{2} \qquad (k\geqq N_0),$$

where N_0 is a certain fixed positive integer. Then we shall have

$$\sum_{i=1}^{n}\alpha_i-\sum_{i=1}^{n}\alpha_i'\geqq \frac{d}{2} \qquad (n\geqq \nu_{N_0}),$$

which is evidently contrary to the hypothesis. This completes the proof of the theorem.

Theorem 2. *Let $\psi(x)$ be a strictly increasing convex function defined on $[0,\infty)$. Then the condition*

$$\text{(N)} \qquad \lim_{n\to\infty}\left(\sum_{i=1}^{n}\psi(x_i)-\sum_{i=1}^{n}\psi(\lambda_i)\right)=0.$$

implies the normalcy of the operator A.

Proof. We have [2]

$$(15)\qquad \begin{array}{l}\psi(\lambda_1)\geqq\psi(\lambda_2)\geqq\cdots, \qquad \psi(x_1)\geqq\psi(x_2)\geqq\cdots,\\ \psi(\lambda_1)+\psi(\lambda_2)+\cdots+\psi(\lambda_\nu)\leqq\psi(x_1)+\psi(x_2)+\cdots+\psi(x_\nu)\quad(\nu=1,2,\ldots).\end{array}$$

Let

$$(16)\qquad \alpha_i'=\psi(\lambda_i)-\psi(0),\quad \alpha_i=\psi(x_i)-\psi(0)\qquad(i=1,2,\ldots).$$

Then the sequence of numbers $\{\alpha_i'\}$ is quasi-majorised by the sequence of numbers $\{\alpha_i\}$, and by theorem I, there exist two sequences of positive integers

$$n_1\leqq n_2\leqq\cdots,\ \nu_1<\nu_2<\cdots \text{ with } n_i\leqq\nu_i \qquad (i=1,2\ldots)$$

so that, for every positive integer k, we can find ν_k^2 non-negative numbers $p_{\mu\nu}^{(k)}\geqq 0$ $(\mu,\nu=1,2,\ldots,\nu_k)$ with

$$\sum_{\nu=1}^{\nu_k}p_{\mu\nu}^{(k)}=1\quad(\mu=1,2,\ldots,\nu_k),\qquad \sum_{\mu=1}^{\nu_k}p_{\mu\nu}^{(k)}=1\qquad(\nu=1,2,\ldots,\nu_k),$$

and

$$(17)\qquad \begin{cases}\alpha_\mu'=\sum_{\nu=1}^{\nu_k}p_{\mu\nu}^{(k)}\alpha_\nu\quad(1\leqq\mu\leqq\nu_k,\quad \mu\neq n_k),\\ \alpha_{n_k}'\leqq\sum_{\nu=1}^{\nu_k}p_{n_k\nu}^{(k)}\alpha_\nu,\\ \lim_{k=\infty}\left(\sum_{\nu=1}^{\nu_k}p_{n_k\nu}^{(k)}\alpha_\nu-\alpha_{n_k}'\right)=0,\end{cases}$$

i. e.

$$(18)\qquad \begin{cases}\psi(\lambda_\mu)=\sum_{\nu=1}^{\nu_k}p_{\mu\nu}^{(k)}\psi(x_\nu)\quad(1\leqq\mu\leqq\nu_k,\quad \mu\neq n_k),\\ \psi(\lambda_{n_k})\leqq\sum_{\nu=1}^{\nu_k}p_{n_k\nu}^{(k)}\psi(x_\nu),\\ \lim_{k=\infty}\left(\sum_{\nu=1}^{\nu_k}p_{n_k\nu}^{(k)}\psi(x_\nu)-\psi(\lambda_{n_k})\right)=0,\end{cases}$$

or

$$(19)\qquad \begin{cases}\psi(\lambda_\mu)=\sum_{\nu=1}^{\nu_k}p_{\mu\nu}^{(k)}\psi(x_\nu)\quad(1\leqq\mu\leqq\nu_k,\quad \mu\neq n_k),\\ \psi(\lambda_{n_k})+\theta_k=\sum_{\nu=1}^{\nu_k}p_{n_k\nu}^{(k)}\psi(x_\nu),\end{cases}$$

where $\theta_k = \sum_{\nu=1}^{\nu_k} p_{n_k\nu}^{(k)} \psi(\varkappa_\nu) - \psi(\lambda_{n_k})$. Hence $\theta_k \geqq 0$, and $\lim_{k=\infty} \theta_k = 0$. Since the inverse function ψ^{-1} of the function ψ is concave and also strictly increasing, we have

(20)
$$\begin{cases} \psi^{-1}\psi(\lambda_\mu) \geqq \sum_{\nu=1}^{\nu_k} p_{\mu\nu}^{(k)} \psi^{-1}\psi(\varkappa_\nu) & (1 \leqq \mu \leqq \nu_k, \quad \mu \neq n_k), \\ \psi^{-1}(\psi(\lambda_{n_k}) + \theta_k) \geqq \sum_{\nu=1}^{\nu_k} p_{n_k\nu}^{(k)} \psi^{-1}\psi(\varkappa_\nu), \end{cases}$$

i. e.

(21)
$$\begin{cases} \lambda_\mu \geqq \sum_{\nu=1}^{\nu_k} p_{\mu\nu}^{(k)} \varkappa_\nu & (1 \leqq \mu \leqq \nu_k, \quad \mu \neq n_k), \\ \lambda_{n_k} + \theta_k' \geqq \sum_{\nu=1}^{\nu_k} p_{n_k\nu}^{(k)} \varkappa_\nu, \end{cases}$$

where $\theta_k' = \psi^{-1}(\psi(\lambda_{n_k}) + \theta_k) - \lambda_{n_k}$. Hence $\theta_k' \geqq 0$ and $\lim_{k\to\infty} \theta_k' = 0$. It follows, by summing up these ν_k inequalities,

(22)
$$\theta_k' + \sum_{\mu=1}^{\nu_k} \lambda_\mu \geqq \sum_{\mu=1}^{\nu_k} \varkappa_\mu,$$

i. e.

(23)
$$\theta_k' \geqq \sum_{\mu=1}^{\nu_k} \varkappa_\mu - \sum_{\mu=1}^{\nu_k} \lambda_\mu.$$

On the other hand

(24)
$$\sum_{\mu=1}^{\nu_k} \varkappa_\mu - \sum_{\mu=1}^{\nu_k} \lambda_\mu \geqq 0.$$

Hence

$$\lim_{k\to\infty} \left(\sum_{\mu=1}^{\nu_k} \varkappa_\mu - \sum_{\mu=1}^{\nu_k} \lambda_\mu \right) = 0.$$

That is to say

(N_1)
$$\lim_{n\to\infty} \left(\sum_{\mu=1}^{n} \varkappa_\mu - \sum_{\mu=1}^{n} \lambda_\mu \right) = 0.$$

Thus the proof of the theorem 2 will be complete, if we are able to prove that the condition (N_1) implies the normalcy of the operator A. For this, we need several lemmas.

Lemma 3. *Let α be an eigenvalue of order ν of the operator A. Then there exists a set of ν orthonormal vectors $\varphi_1, \varphi_2, \ldots, \varphi_\nu$, so that*

$$A\varphi_\mu = \sum_{\nu=1}^{\mu} \alpha_{\mu j} \varphi_j \qquad (\mu = 1, 2, \ldots, \nu)$$

with $\alpha_{11} = \alpha_{22} = \cdots = \alpha_{\nu\nu} = \alpha$.

Lemma 4. *Let $\xi_1, \xi_2, \ldots, \xi_m$ be a set of m orthonormal vectors and*

$$A\xi_\mu = \sum_{\nu=1}^{\mu} \beta_{\mu\nu}\xi_\nu \qquad (\mu = 1, 2, \ldots, m).$$

Suppose γ is an eigenvalue of order n of the operator A, and that the characteristic function of the matrix

$$\begin{pmatrix} \beta_{11} & & & \\ \beta_{21} & \beta_{22} & & \\ \cdot & \cdot & \cdot & \\ \beta_{m1} & \beta_{m2} & \ldots & \beta_{mm} \end{pmatrix}$$

and $(x-\gamma)^n$ are relatively prime. Then there exist n vectors $\eta_1, \eta_2, \ldots, \eta_n$, so that

1) *$\xi_1, \xi_2, \ldots, \xi_m, \eta_1, \eta_2, \ldots, \eta_n$ is a set of $m+n$ orthonormal vectors,*

2) *$A\eta_i = \sum_{j=1}^{m+i} \nu_{m+i,\,j}\,\eta_j \quad (i = 1, 2, \ldots, n)$ with*

$$\nu_{m+1,\,m+1} = \nu_{m+2,\,m+2} = \cdots = \nu_{m+n,\,m+n} = \gamma.$$

By lemmas 3, 4, there exists an orthonormal system of vectors $\omega_1, \omega_2, \ldots$, such that the closed linear manifold $\mathfrak{M}$ generated from these vectors is invariant under the transformation A, and

$$A\omega_i = \alpha_{i1}\omega_1 + \alpha_{i2}\omega_2 + \cdots + a_{ii}\omega_i \qquad (i = 1, 2, \ldots) \tag{25}$$

with $a_{ii} = \alpha_i \quad (i = 1, 2, \ldots)$. But

$$\sum_{i=1}^{n} (A^*A\omega_i, \omega_i) \leqq \sum_{i=1}^{n} \varkappa_i \qquad (n = 1, 2, \ldots),$$

i. e.

$$\sum_{i=1}^{n} (A\omega_i, A\omega_i) \leqq \sum_{i=1}^{n} \varkappa_i \qquad (n = 1, 2, \ldots),$$

or

$$\sum_{\substack{i \geqq j \\ i,j=1,2,\ldots,n}} |a_{ij}|^2 \leqq \sum_{i=1}^{n} \varkappa_i \qquad (n = 1, 2, \ldots),$$

or

$$\sum_{\substack{i > j \\ i,j=1,2,\ldots,n}} |a_{ij}|^2 \leqq \sum_{i=1}^{n} \varkappa_i - \sum_{i=1}^{n} \lambda_i \qquad (n = 1, 2, \ldots). \tag{26}$$

From (26) and the condition (N_1) it results immediately

$$\sum_{i > j} |a_{ij}|^2 = 0.$$

Hence $a_{ij} = 0$ $(i > j;\ i, j = 1, 2, \ldots)$ and

$$A\omega_i = \alpha_i\omega_i \qquad (i = 1, 2, \ldots).$$

Let $\mathfrak{N}$ be the orthogonal complement of $\mathfrak{M}$. If ω is any vector of unit norm in $\mathfrak{N}$, then we have

$$\sum_{i=1}^{n}(A^*A\omega_i, \omega_i)+(A^*A\omega, \omega) \leqq \sum_{i=1}^{n+1}\varkappa_i \qquad (n=1,2,\ldots),$$

i. e.

$$\sum_{i=1}^{n}\lambda_i+(A^*A\omega, \omega) \leqq \sum_{i=1}^{n+1}\varkappa_i \qquad (n=1,2,\ldots),$$

or

$$(A^*A\omega, \omega) \leqq \left(\sum_{i=1}^{n}\varkappa_i-\sum_{i=1}^{n}\lambda_i\right)+\varkappa_{n+1} \qquad (n=1,2,\ldots). \tag{27}$$

Condition (N_1) and (27) together with that $\lim_{n\to\infty}\varkappa_n=0$ imply

$$(A^*A\omega, \omega)=0; \quad (A\omega, A\omega)=0.$$

Hence $A\omega=0$ for every $\omega\in\mathfrak{N}$. This proves the normalcy of the operator A.

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We conclude this paper by giving an example to show that theorem 2 is no longer true, if the function $\psi(\varkappa)$ is supposed to be strictly increasing on $[0,\infty)$, so that $\psi(e^\xi)$ is a convex function of ξ. Let

$$\psi(\varkappa)=\begin{cases} \varkappa, & (0\leqq\varkappa\leqq 1),\\ \dfrac{\log\varkappa}{\log 2}+1 & (1<\varkappa<\infty).\end{cases}$$

Then

$$\psi(e^\xi)=\begin{cases} e^\xi & (-\infty<\xi\leqq 0),\\ \dfrac{\xi}{\log 2}+1 & (0\leqq\xi<\infty).\end{cases}$$

Evidently $\psi(\varkappa)$ is a strictly increasing function defined on $[0,\infty)$, and $\psi(e^\xi)$ is a convex function of ξ. Suppose A is linear operator in the 2-dimensional Euclidean space. Using matrix notation, we let

$$A=\begin{pmatrix}\sqrt{2} & 1\\ & \sqrt{2}\end{pmatrix}.$$

Thus

$$A^*=\begin{pmatrix}\sqrt{2} & \\ 1 & \sqrt{2}\end{pmatrix} \qquad A^*A=\begin{pmatrix}2 & \sqrt{2}\\ \sqrt{2} & 3\end{pmatrix}.$$

The eigenvalues of A are $\alpha_1=\alpha_2=\sqrt{2}$ (hence $\lambda_1=\lambda_2=2$) and the eigenvalues of A^*A are $\varkappa_1=4$, $\varkappa_2=1$. Now

$$\psi(\lambda_1)+\psi(\lambda_2)=2\psi(2)=2\left(\frac{\log 2}{\log 2}+1\right)=4,$$

$$\psi(\varkappa_1)+\psi(\varkappa_2)=\psi(4)+\psi(1)=\left(\frac{\log 4}{\log 2}+1\right)+1=4.$$

Hence $\psi(\lambda_1)+\psi(\lambda_2)=\psi(\varkappa_1)+\psi(\varkappa_2)$. But A is not normal. This gives a counter example.

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