Proof of a conjecture of P. Erdős.

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Let E be a given non countable set of power $\mathfrak{m}$ and suppose that there exists a relation[1] R between the elements of E such that, for any $x \in E$, the power of the set $H(x)$ of the elements $y \in E$ $(y \neq x)$ for which xRy holds, is smaller than a given cardinal number $\mathfrak{n}$ which is smaller than $\mathfrak{m}$. Two distinct elements (or "points") x and y of E are called independent if neither xRy nor yRx. We say that a subset of E is a *free set* if any two points of this subset are independent.

If we replace the condition $\mathfrak{n} < \mathfrak{m}$ by $\mathfrak{n} \leqq \mathfrak{m}$ then it can occur that we do not have any independent points at all. Indeed, let φ be the initial number of power $\mathfrak{m}$ and E the set of ordinal numbers less than φ. We define the relation R so that xRy holds if and only if $y < x$. Then clearly $\overline{\overline{H(x)}} < \mathfrak{m}$ for any $x \in E$; however, no two elements are independent.

The following proposition has been conjectured by S. RUZIEWICZ[2]:

If $\mathfrak{n} < \mathfrak{m}$, *then* E *has a free subset* E^* *of the same power* $\mathfrak{m}$.

This theorem has been proved first if $\mathfrak{n} = \aleph_0$ and $\mathfrak{m}$ is either of the form $2^{\mathfrak{p}}$ or of the form $\aleph_{\alpha+1}$[3]), then if $\mathfrak{m}$ is a regular cardinal number or if $\mathfrak{m}$ is the countable sum of cardinals smaller than $\mathfrak{m}$[4]), finally, in the general case, assuming the generalized continuum hypothesis[5]).

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1) "Relation" means throughout this paper a binary relation.

2) S. RUZIEWICZ, Une généralisation d'un théorème de M. Sierpiński, *Publications Math. de l'Université de Belgrade,* 5 (1936), pp. 23—27.

3) W. SIERPIŃSKI, Sur un problème de la théorie des relations, *Fundamenta Math.*, 28 (1937), pp. 71—74. — D. LÁZÁR, On a problem in the theory of aggregates, *Compositio Math.*, 3 (1936), 304.

4) SOPHIE PICCARD, Sur un problème de M. Ruziewicz de la théorie des relations, *Fundamenta Math.*, 29 (1937), pp. 5—9; Solution du problème de M. Ruziewicz de la théorie des relations pour les nombres cardinaux $\mathfrak{m} < \aleph_\Omega$, *Comptes Rendus Varsovie,* 30 (1937), pp. 12—18.

5) P. ERDŐS, Some remarks on set theory, *Proceedings American Math. Soc.*, 1 (1950), pp. 133—137.

The proof given by SIERPIŃSKI[8]) yields also the fact that, if $\mathfrak{n}=\aleph_0$ and $\mathfrak{m}=2^{\mathfrak{p}}$, then E is the union of $\mathfrak{p}$ free subsets. The proof of LÁZÁR[2]) yields the same fact in the case $\mathfrak{n}=\aleph_0$, $\mathfrak{m}=2^{\aleph_0}$.

DE BRUIJN and ERDŐS[6]) proved for a set E of arbitrary power the following statements: If for every $x\in E$, the set $H(x)$ has at most k elements, k being a given positive integer, then E may be decomposed in $2k+1$ or fewer free sets; if for every $x\in E$ the set $H(x)$ is finite, then E is the union of a countable number of free sets.

We shall now prove the following theorem which was conjectured by ERDŐS[5]):

Theorem 1. *If E is a non countable set of power $\mathfrak{m}$ and if R is a relation between the elements of E such that for any $x\in E$ the power of the set $H(x)$ of the elements $y\in E$ $(y\neq x)$ for which xRy holds is smaller than a given cardinal number $\mathfrak{n}$, where $\aleph_0\leqq\mathfrak{n}<\mathfrak{m}$, then E may be decomposed into the sum of $\mathfrak{n}$ or fewer free subsets.*

As a consequence of this theorem, we see at once that the conjecture of RUZIEWICZ holds if $\mathfrak{m}$ cannot be decomposed into a sum of $\mathfrak{n}$ or fewer cardinal numbers, each of which is smaller than $\mathfrak{m}$.

§ 1.

First we prove the following theorem.

Theorem 2. *Let $\mathfrak{n}$ be a regular transfinite cardinal number, ψ the initial number of the cardinal number $\mathfrak{n}$ and E an arbitrary set. Suppose that a relation R is defined between the elements of E such that the set $H(x)$ of the elements $y\in E$ $(y\neq x)$ for which xRy holds, has a cardinal number smaller than $\mathfrak{n}$. Then E can be well-ordered into a transfinite sequence*

$$p_0, p_1, p_2, \ldots, p_\omega, p_{\omega+1}, \ldots, p_\xi, \ldots \qquad (\xi<\alpha) \tag{1}$$

in such a way that we have

$$\sum_{\zeta<\psi\mu} H(p_\zeta)\subseteqq\{p_\zeta\}_{\zeta<\psi\mu} \tag{2}$$

for every μ, $1\leqq\mu<\gamma$, where γ is defined by the equality $\alpha=\psi\gamma+\delta$ $(\delta<\psi)$.

Proof. Let φ be the initial number of the cardinal number $\overline{\overline{E}}=\mathfrak{m}$ and let

$$x_0, x_1, x_2, \ldots, x_\omega, x_{\omega+1}, \ldots, x_\xi, \ldots \qquad (\xi<\varphi) \tag{3}$$

be any well-ordering of E of the type φ. We define the sequence (1) by transfinite induction in the following way: Put $p_0=x_0$. Let now β be an ordinal number, $\beta>0$, and suppose that all elements p_ζ, where $0\leqq\zeta<\beta$,

[6]) N. G. DE BRUIJN and P. ERDŐS, A colour problem for infinite graphs and a problem in the theory of relations, *Proceedings Amsterdam*, 54 (1951), pp. 371—372.

have been already defined and let P_β denote the set of the elements p_ζ with $\zeta < \beta$. Consider the set

$$V_\beta = \sum_{\zeta<\beta} H(p_\zeta).$$

If $V_\beta \neq 0$, we define a new well-ordering of V_β as follows. Let q and r be any two distinct elements of V_β. Let $\varkappa$ and λ be the least ordinals for which $q \in H(p_\varkappa)$ and $r \in H(p_\lambda)$, respectively. Write $q \prec r$ if either $\varkappa < \lambda$ or if $\varkappa = \lambda$ but q precedes r in $H(p_\varkappa)$ in the original well-ordering (3) (as a subset of E which is well-ordered according to (3)). In the sequel we suppose always that V_β is well-ordered in this way. Let

$$W_\beta = V_\beta - P_\beta.$$

(i) If $W_\beta \neq 0$, let p_β be the first element of W_β (as a subset of V_β).

(ii) If $W_\beta = 0$ and $P_\beta \neq E$, let p_β the first element of $E - P_\beta$ (in the well-ordering (3)).

(iii) If $W_\beta = 0$ and $P_\beta = E$, then we do not define p_β.

Clearly, case (iii) occurs for one and only one value α of β; for $\beta < \alpha$, p_β, V_β and W_β are defined. For $\nu < \pi < \alpha$, the set V_ν is obviously a section of V_π.

Next we prove the following

L e m m a. *Suppose $\beta < \alpha$ and $W_\beta \neq 0$. Let*

$$w_0, w_1, w_2, \ldots, w_\omega, w_{\omega+1}, \ldots, w_\xi, \ldots \qquad (\xi < \overline{W_\beta})$$

be the well-ordering of the set W_β (as a subset of V_β). Then we have $p_{\beta+\xi} = w_\xi$ for $\xi < \overline{W_\beta}$.

Indeed, this holds by definition for $\xi = 0$. Suppose, our statement holds for any ordinal number which is smaller than ξ $(< \overline{W_\beta})$; then it holds for ξ too. Indeed, $p_{\beta+\xi}$ is, by definition, the first element of $W_{\beta+\xi}$. Now we have $w_\xi \in W_\beta \subseteq V_\beta \subset V_{\beta+\xi}$, hence $w_\xi \in V_{\beta+\xi}$. On the other hand, $w_\xi \notin P_\beta$ and, by hypothesis, $P_{\beta+\xi} = P_\beta + \{w_\eta\}_{\eta<\xi}$; hence $w_\xi \notin P_{\beta+\xi}$. Therefore $w_\xi \in W_{\beta+\xi}$. Further, any element of $W_{\beta+\xi}$ preceding w_ξ is an element of $V_{\beta+\xi}$ preceding w_ξ, hence an element of V_β preceding w_ξ for V_β is a section of $V_{\beta+\xi}$ and $w_\xi \in V_\beta$. Now, any element of V_β preceding w_ξ is either an element of P_β or an element of W_β preceding w_ξ; hence in any case an element of $P_{\beta+\xi}$. Therefore, such an element cannot belong to $W_{\beta+\xi} = V_{\beta+\xi} - P_{\beta+\xi}$. Hence, w_ξ is the first element of $W_{\beta+\xi}$, thus $p_{\beta+\xi} = w_\xi$ as stated.

Now we prove by transfinite induction that (2) holds for every μ, $1 \leqq \mu < \gamma$. This is obvious for $\mu = 0$. Suppose (2), i. e. $V_{\psi\mu} \subseteq P_{\psi\mu}$ holds for some μ; then we prove the same for $\mu+1$ instead of μ, i. e.

$$V_{\psi(\mu+1)} = \sum_{\zeta<\psi(\mu+1)} H(p_\zeta) \subseteq P_{\psi(\mu+1)}.$$

As we have, by hypothesis,

$$V_{\psi\mu} = \sum_{\zeta < \psi\mu} H(p_\zeta) \subseteq P_{\psi\mu} \subseteq P_{\psi(\mu+1)},$$

we have to prove that

$$\sum_{\psi\mu \leqq \zeta < \psi(\mu+1)} H(p_\zeta) \subseteq P_{\psi(\mu+1)}. \tag{4}$$

For this purpose, let ζ be an ordinal number such that $\zeta = \psi\mu + \varrho$ with $\varrho < \psi$, and denote by h any element of $H(p_\zeta)$. By the definition of V_β, we have $h \in V_{\zeta+1}$. If $h \in P_{\zeta+1}$, then we have $h \in P_{\psi(\mu+1)}$, for, $\psi(\mu+1)$ being an ordinal number of the second kind, $\zeta+1 < \psi(\mu+1)$. If $h \notin P_{\zeta+1}$ then we have, by the definition of W_β, $h \in W_{\zeta+1}$. Applying the lemma with $\beta = \zeta+1$, we see that $h = p_{\zeta+1+\xi}$ for some $\xi < \overline{W}_{\zeta+1}$. Now we have

$$W_{\zeta+1} = V_{\zeta+1} - P_{\zeta+1} = \sum_{\eta < \zeta+1} H(p_\eta) - P_{\zeta+1} \subseteq \sum_{\psi\mu \leqq \eta < \zeta+1} H(p_\eta)$$

for, by the induction hypothesis, any element of

$$\sum_{\eta < \psi\mu} H(p_\eta) = V_{\psi\mu}$$

belongs to $P_{\psi\mu}$ and thus to $P_{\zeta+1}$. Therefore we have

$$\overline{\overline{W_{\zeta+1}}} \leqq \overline{\overline{\sum_{\psi\mu \leqq \eta < \zeta+1} H(p_\eta)}} \leqq \sum_{\psi\mu \leqq \eta < \zeta+1} \overline{\overline{H(p_\eta)}} = \sum_{\tau < \varrho+1} \overline{\overline{H(p_{\psi\mu+\tau})}} < \mathfrak{n},$$

because $\overline{\overline{H(p_\eta)}} < \mathfrak{n}$ for any η and $\overline{\overline{\varrho+1}} = \overline{\overline{\varrho}} < \overline{\overline{\psi}} = \mathfrak{n}$, and $\mathfrak{n}$ is regular. Hence we have $\xi < \overline{\overline{W}}_{\zeta+1} < \psi$ and consequently $\zeta + 1 + \xi = \psi\mu + \varrho + 1 + \xi < \psi\mu + \psi = \psi(\mu+1)$, i. e. $h = p_{\zeta+1+\xi} \in P_{\psi(\mu+1)}$ in this case too, which proves (4).

Let now μ be an ordinal number of the second kind, $\mu < \gamma$. Suppose that

$$V_{\psi\nu} \subseteq P_{\psi\nu}$$

for every ordinal number $\nu < \mu$. We have to prove that $V_{\psi\mu} = \sum_{\zeta < \psi\mu} H(p_\zeta) \subseteq P_{\psi\mu}$. For this purpose let ζ be any ordinal number satisfying $\zeta < \psi\mu$. This inequality implies $\zeta < \psi\nu$ for some $\nu < \mu$, for μ is an ordinal number of the second kind. Hence, any element of $H(p_\zeta)$ belongs to $\sum_{\eta < \psi\nu} H(p_\eta) = V_{\psi\nu}$, thus, by the induction hypothesis, also to $P_{\psi\nu}$, hence to $P_{\psi\mu}$ too, which proves our statement. Hence, theorem 2 is proved.

§ 2.

By means of Theorem 2 we prove the following theorem:

Theorem 3. *Let $\mathfrak{n}$ be a regular transfinite cardinal number and E an arbitrary set; further let R be a relation defined between the elements of E such that the set $H(x)$ of the elements $y \in E$ $(y \neq x)$ for which xRy holds has a power smaller then $\mathfrak{n}$. Then there exists a system $X = \{F_\eta\}$ of* *mutually disjoint free subsets* F_η *of* E *such that* $\overline{\overline{X}} \leqq \mathfrak{n}$ *and that for any element* y *of* $E - \sum_{F_\eta \in X} F_\eta$ *there is an element* $x \in \sum_{F_\eta \in X} F_\eta$ *for which* yRx *holds.*

Proof. Denote again by ψ the initial number of the cardinal number $\mathfrak{n}$. Applying theorem 2, we obtain a transfinite sequence (1) for which (2) holds (for every μ, $1 \leqq \mu < \gamma$, γ being defined as above). Let Q_μ denote the set of the elements p_ζ with $\psi\mu \leqq \zeta < \psi(\mu+1)$ for $0 \leqq \mu < \gamma$ and, for $\mu = \gamma$, the set of the elements p_ζ with $\psi\gamma \leqq \zeta < \alpha$. Obviously, the sets Q_μ are mutually disjoint and we have $\sum_{\mu \leqq \gamma} Q_\mu = E$.

Let $Z(x)$ denote, for every $x \in E$, the set of $y \in E$ $(y \neq x)$ for which yRx holds; further, let $Z[F]$ denote, for every $F \subseteqq E$ the set $\sum_{x \in F} Z(x)$.

First we define the set F_0. Let $f_{00} = p_0$. Let λ be a given ordinal number, $\lambda \geqq 1$, and suppose that $f_{0\varkappa}$ $(\in E)$ is defined for every $\varkappa < \lambda$. The condition $f_{0\varkappa} \in Q_{\mu_{0\varkappa}}$ defines uniquely an ordinal number $\mu_{0\varkappa}$. If there is an ordinal number μ which is greater then every $\mu_{0\varkappa}$ $(\varkappa < \lambda)$ for which Q_μ is not a subset of $\sum_{\varkappa < \lambda} Z(f_{0\varkappa})$ then let μ' be the smallest such ordinal number and define $f_{0\lambda}$ as the first element of $Q_{\mu'} - \sum_{\varkappa < \lambda} Z(f_{0\varkappa})$ in the well-ordering (1). Clearly, we have $\mu_{0\lambda} = \mu'$. In the opposite case, i. e. if $Q_\mu \subseteqq \sum_{\varkappa < \lambda} Z(f_{0\varkappa})$ for any $\mu > \mu_{0\varkappa}$ $(\varkappa < \lambda)$, then we do not define $f_{0\lambda}$. We define F_0 as the set of all those $f_{0\lambda}$ which have been defined.

Let η be a given ordinal number, $\eta \geqq 1$, and suppose that the subset F_ζ of E is defined for every $\zeta < \eta$. Supposing that the set

$$A_\eta = \sum_{\zeta < \eta} (F_\zeta \dot{+} Z[F_\zeta])$$

is a proper subset of E, we define the subset F_η of E as follows. Let $\mu_{\eta 0}$ be the smallest ordinal number μ for which Q_μ is not a subset of A_η. (There exists such an ordinal number Q_μ, for $A_\eta \neq E$.) Define $f_{\eta 0}$ as the first element of $Q_{\mu_{\eta 0}} - A_\eta$ in the well-ordering (1). Let λ be an arbitrary ordinal number, $\lambda \geqq 1$, and suppose the element $f_{\eta\varkappa}$ of $E - A_\eta$ is defined for every $\varkappa < \lambda$. Define $\mu_{\eta\varkappa}$ for $\varkappa < \lambda$ by the condition $f_{\eta\varkappa} \in Q_{\nu_k}$. (For $\varkappa = 0$, this agrees with the above definition of $\mu_{\eta 0}$.) If there is an ordinal number μ which is greater than every $\mu_{\eta\varkappa}$ $(\varkappa < \lambda)$ for which Q_μ is not a subset of $A_\eta + \sum_{\varkappa < \lambda} Z(f_{\eta\varkappa})$, then let μ' be the smallest such ordinal number and define $f_{\eta\lambda}$ as the first element of $Q_{\mu'} - (A_\eta + \sum_{\varkappa < \lambda} Z(f_{\eta\varkappa}))$ in the well-ordering (1). Clearly, we have $\mu_{\eta\lambda} = \mu'$. In the opposite case, i. e. if $Q_\mu \subset A_\eta + \sum_{\varkappa < \lambda} Z(f_{\eta\varkappa})$ for any $\mu > \mu_{\eta\varkappa}$ $(\varkappa < \lambda)$, then we do not define $f_{\eta\lambda}$. We define F_η as the set of all those $f_{\eta\lambda}$ which have been defined. If, however, we have $A_\eta = E$, then we do not define the set F_η. Finally, we define X as the set of all those F_η which have been defined.

As an immediate consequence of this definition, we see that the elements F_η of X are mutually disjoint subsets of E. We prove first that they are free sets. Indeed, any two distinct elements of F_η are of the form $f_{\eta\varkappa}$ and $f_{\eta\lambda}$ ($\varkappa \neq \lambda$). Let $\varkappa < \lambda$, say. Then, by the definition, we have $f_{\eta\lambda} \in Q_{\mu_{\eta\lambda}} - (A_\eta + \sum_{\varkappa<\lambda} Z(f_{\eta\varkappa}))$ (also in case $\eta = 0$, for then we have $f_{\eta\lambda} = f_{0\lambda} \in Q_{\mu_{0\lambda}} - \sum_{\varkappa<\lambda} Z(f_{0\varkappa}) = Q_{\mu_{\eta\lambda}} - (A_\eta + \sum_{\varkappa<\lambda} Z(f_{\eta\varkappa}))$ on account of $A_0 = 0$). Hence $f_{\eta\lambda} \notin Z(f_{\eta\varkappa})$, i. e. $f_{\eta\lambda} R f_{\eta\varkappa}$ does not hold. On the other hand, we have $f_{\eta\lambda} \in Q_{\mu_{\eta\lambda}}$ and $f_{\eta\varkappa} \in Q_{\mu_{\eta\varkappa}}$, and here $\mu_{\eta\lambda} > \mu_{\eta\varkappa}$. Hence, by the definition of the sets Q_μ, we have $f_{\eta\varkappa} = p_\zeta$ and $f_{\eta\lambda} = p_\xi$ for some ζ and ξ,

$$\psi\mu_{\eta\varkappa} \leqq \zeta < \psi(\mu_{\eta\varkappa} + 1) \leqq \psi\mu_{\eta\lambda} \leqq \xi < \psi(\mu_{\eta\lambda} + 1).$$

Hence, by (2) we have $H(f_{\eta\varkappa}) = H(p_\zeta) \subseteq P_{\psi(\mu_{\eta\varkappa}+1)}$ whereas we have $f_{\eta\lambda} = p_\xi \notin P_{\psi(\mu_{\eta\varkappa}+1)}$ for $\xi \geqq \psi(\mu_{\eta\varkappa} + 1)$. Hence $f_{\eta\lambda} \notin H(f_{\eta\varkappa})$, i. e. $f_{\eta\varkappa} R f_{\eta\lambda}$ does not hold either. Thus, any two elements $f_{\eta\varkappa}$ and $f_{\eta\lambda}$ of F_η are independent, i. e. F_η is indeed a free set.

Next we prove $\overline{\overline{X}} \leqq \mathfrak{n}$. For this purpose, it is sufficient to show that, for any $F_\eta \in X$, we have $\eta < \psi$. This is obvious for $\eta = 0$. Suppose $F_\eta \in X$, i. e. that F_η has been defined and $\eta \neq 0$. Then $Q_{\mu_{\eta 0}}$ is not a subset of $A_\eta = \sum_{\zeta<\eta} (F_\zeta + Z[F_\zeta])$. Every set $A_\zeta = \sum_{\xi<\zeta} (F_\xi + Z[F_\xi])$ ($\zeta < \eta$) and, moreover, every set $A_\zeta + \sum_{\varkappa<\lambda} Z(f_{\zeta\varkappa}) = \sum_{\xi<\zeta} (F_\xi + Z[F_\xi]) + \sum_{\varkappa<\lambda} Z(f_{\zeta\varkappa})$ (where $\zeta < \eta$ and $f_{\zeta\varkappa} \in F_\zeta$ for any $\varkappa < \lambda$) being obviously a subset of A_η, $Q_{\mu_{\eta 0}}$ is not a subset of any such set A_ζ or $A_\zeta + \sum_{\varkappa<\lambda} Z(f_{\zeta\varkappa})$. Hence, for a suitable λ ($f_{\zeta\lambda} \in F_\zeta$) we have $\mu_{\eta 0} = \mu_{\zeta\lambda}$. Indeed, in the opposite case we would have $\mu_{\zeta\lambda} < \mu_{\eta 0}$ for every λ, $f_{\zeta\lambda} \in F_\zeta$. This is obvious for $\lambda = 0$, because $\mu_{\zeta 0}$ is, by definition, the smallest ordinal number μ for which Q_μ is not a subset of A_ζ, and $\mu_{\eta 0}$ is such an ordinal number. Suppose, we have $\mu_{\zeta\varkappa} < \mu_{\eta 0}$ for every $\varkappa < \lambda$. Then we have also $\mu_{\zeta\lambda} < \mu_{\eta 0}$, for $\mu_{\zeta\lambda}$ is by definition the smallest ordinal number μ for which $\mu > \mu_{\zeta\varkappa}$ ($\varkappa < \lambda$) and for which Q_μ is not a subset of $A_\zeta + \sum_{\varkappa<\lambda} Z(f_{\zeta\varkappa})$, and this holds for the ordinal number $\mu = \mu_{\eta 0}$ too. Now, let λ be the smallest ordinal number for which $f_{\zeta\lambda}$ has not been defined. Then we have $Q_\mu \subseteq A_\zeta + \sum_{\varkappa<\lambda} Z(f_{\zeta\varkappa})$ for any μ which is greater then any $\mu_{\zeta\varkappa}$ ($\varkappa < \lambda$); but this impossible since it does not hold for $\mu = \mu_{\eta 0}$.

Thus we have $f_{\zeta\lambda} \in Q_{\mu_{\zeta\lambda}} = Q_{\mu_{\eta 0}}$ for any $\zeta < \eta$ and for a suitable $\lambda = \lambda(\zeta) = \lambda(\zeta, \eta)$. This holds also for $\zeta = \eta$ with $\lambda = \lambda(\eta) = 0$. Now we prove that for $\xi < \zeta \leqq \eta$ we have $f_{\xi\lambda(\xi)} \prec f_{\zeta\lambda(\zeta)}$ in the well-ordering (1). Indeed, $f_{\xi\lambda(\xi)}$

is by definition the first element of

$$Q_{\mu_\xi \lambda(\xi)} - (A_\xi + \sum_{\varkappa < \lambda(\xi)} Z(f_{\xi\varkappa})) = Q_{\mu_\eta 0} - (A_\xi + \sum_{\varkappa < \lambda(\xi)} Z(f_{\xi\varkappa})).$$

On the other hand, we have $f_{\zeta\lambda(\zeta)} \in Q_{\mu_\zeta \lambda(\zeta)} - (A_\zeta + \sum_{\varkappa < \lambda(\zeta)} Z(f_{\zeta\varkappa}))$. Hence we have $f_{\zeta\lambda(\zeta)} \in Q_{\mu_\zeta \lambda(\zeta)} = Q_{\eta 0}$ and $f_{\zeta\lambda(\zeta)} \notin A_\zeta = \sum_{\varrho < \zeta} (F_\varrho + Z[F_\varrho])$. Now obviously

$$A_\xi + \sum_{\varkappa < \lambda(\xi)} Z(f_{\xi\varkappa}) = \sum_{\varrho < \xi} (F_\varrho + Z[F_\varrho]) + \sum_{\varkappa < \lambda(\xi)} Z(f_{\xi\varkappa}) \subseteq A_\zeta;$$

hence $f_{\zeta\lambda(\zeta)} \notin A_\xi + \sum_{\varkappa < \lambda(\xi)} Z(f_{\xi\varkappa})$. Consequently, we have

$$f_{\zeta\lambda(\zeta)} \in Q_{\mu_\eta 0} - (A_\xi + \sum_{\varkappa < \lambda(\xi)} Z(f_{\xi\varkappa}));$$

hence, $f_{\xi\lambda(\xi)}$ being the first element of this set, we have $f_{\xi\lambda(\xi)} \preceq f_{\zeta\lambda(\zeta)}$. By the disjointness of the sets F_ξ and F_ζ, this implies $f_{\xi\lambda(\xi)} \prec f_{\zeta\lambda(\zeta)}$ as stated.

Hence the elements $f_{\zeta\lambda(\zeta)}$ $(\zeta < \eta)$ form a subset of $Q_{\mu_\eta 0}$ which is similar to the set of the ordinal numbers ζ $(\zeta < \eta)$. On the other hand, on account of $f_{\zeta\lambda(\zeta)} \prec f_{\eta\lambda(\eta)} = f_{\eta 0}$ this subset is a subset of the section of $Q_{\mu_\eta 0}$ formed by the element $f_{\eta 0}$. Thus the ordinal number of this subset is smaller then the ordinal number of $Q_{\mu_\eta 0}$, hence smaller than ψ. The set of the ordinal numbers $\zeta (\zeta < \eta)$ having the ordinal number η, we see that $\eta < \psi$, and hence $\overline{X} \leqq \mathfrak{n}$ indeed.

We have yet to prove that for any element y of $E - \sum_{F_\eta \in X} F_\eta$ there is an element x of $\sum_{F_\eta \in X} F_\eta$ for which yRx holds. Indeed, let τ denote the smallest ordinal number for which F_τ has not been defined. Then we have

$$E = A_\eta = \sum_{\eta < \tau} (F_\eta + Z[F_\eta]).$$

Hence

$$E - \sum_{F_\eta \in X} F_\eta = E - \sum_{\eta < \tau} F_\eta \subseteq \sum_{\eta < \tau} Z[F_\eta] = Z[\sum_{\eta < \tau} F_\eta]$$

which shows, that for any $y \in E - \sum_{F_\eta \in X} F_\eta$ we have $y \in Z[\sum_{\eta < \tau} F_\eta]$, i. e. $y \in Z(x)$, that is, yRx for a suitable $x \in \sum_{\eta < \tau} F_\eta = \sum_{F_\eta \in X} F_\eta$, as stated. Thus Theorem 3 has been proved.

§ 3.

Now we can prove Theorem 1 for any regular transfinite cardinal number $\mathfrak{n}$. Indeed, suppose the set E and the relation R satisfy the conditions of Theorem 1. Define the sets E_α and X_α by transfinite induction as follows. Let $E_0 = E$ be and X_0 the system X belonging to the set E_0, satisfying the statement of Theorem 3. Suppose, α is an ordinal number such that for any ordinal number $\beta < \alpha$, the subset E_β of E and the system X_β of some subsets of E_β have been defined. If $\sum_{\beta<\alpha}\sum_{F\in X_\beta} F$ is a proper subset of E, then we put

$$E_\alpha = E - \sum_{\beta<\alpha}\sum_{F\in X_\beta} F$$

and we define X_α as the system X corresponding in the sense of Theorem 3 to the set E_α (instead of E). (Obviously, any subset E_α of E satisfies the conditions of Theorem 3.) If, however, $\sum_{\beta<\alpha}\sum_{F\in X_\beta} F = E$ then we do not define E and X_α.

Now we prove that if E_α is defined (and therefore, by the definition, non empty), then for any $y \in E_\alpha$ and $\beta < \alpha$, there exists an element $x = x(\beta) \in \sum_{F\in X_\beta} F$ such that we have yRx. This holds (vacuously) for $\alpha = 0$. Suppose $\alpha \geqq 1$ and that the statement holds for any $\beta < \alpha$; than we prove the same for α. Indeed, let $\beta < \alpha$. Suppose first that there is an ordinal number γ for which $\beta < \gamma < \alpha$. Then we have obviously $E_\alpha \subseteq E_\gamma$ hence $y \in E_\alpha$ implies $y \in E_\gamma$ and thus, by hypothesis, the existence of an $x \in \sum_{F\in X_\beta} F$ for which yRx, as stated. If, on the contrary, no such ordinal number γ exists, then we have $\alpha = \beta + 1$, thus

$$E_\alpha = E - \sum_{\xi<\beta+1}\sum_{F\in X_\xi} F = E - \sum_{\xi<\beta}\sum_{F\in X_\xi} F - \sum_{F\in X_\beta} F = E_\beta - \sum_{F\in X_\beta} F.$$

Now, by Theorem 3, for any $y \in E_\beta - \sum_{F\in X_\beta} F = E_\alpha$ there is an $x \in \sum_{F\in X_\beta} F$ for which we have yRx, so that our statement holds in this case too.

Now, the sets $\sum_{F\in X_\alpha} F$ are mutually disjoint. Indeed, if $\beta < \alpha$, then X_α is, by definition, a system of subsets of E_α, thus $\sum_{F\in X_\alpha} F$ is a subset of $E_\alpha = E - \sum_{\beta<\alpha}\sum_{F\in X_\beta} F$, hence has no element in common with $\sum_{F\in X_\beta} F$. Therefore, if E_α is defined and thus not empty, and if y is an arbitrary element of E_α, then the set of the above elements $x(\beta)$ $(\beta < \alpha)$ has the cardinal number $\bar{\alpha}$. On the other hand, we have $yRx(\beta)$ i. e. $x(\beta) \in H(y)$ for any $\beta < \alpha$. This implies $\bar{\alpha} \leqq \overline{\overline{H(y)}} < \mathfrak{n}$.

Hence, there exists a least ordinal number α with $\bar{\alpha} \leqq \mathfrak{n}$ for which E_α is not defined, therefore

$$E = \sum_{\beta<\alpha}\sum_{F\in X_\beta} F. \tag{4}$$

By Theorem 3 we have $X_\beta \leqq \mathfrak{n}$ for any $\beta < \alpha$, thus (4) furnishes a decomposition of E into a sum of at most $\mathfrak{n}\cdot\mathfrak{n} = \mathfrak{n}$ free subsets, which proves theorem 1 in the case that $\mathfrak{n}$ is a regular cardinal number.

§ 4.

We assume now that $\mathfrak{n}$ is a singular cardinal number. Let $\mathfrak{r}$ denote the smallest cardinal number such that $\mathfrak{n}$ is the sum of $\mathfrak{r}$ cardinal numbers each of which is less than $\mathfrak{n}$. Since $\mathfrak{n}$ is singular, we have $\mathfrak{r} < \mathfrak{n}$. Let μ denote the initial number of $\mathfrak{r}$. There exist regular cardinal numbers $\mathfrak{n}_1, \mathfrak{n}_2, \ldots, \mathfrak{n}_\varkappa, \ldots$ $(\varkappa < \mu)$ such that $\mathfrak{n}_\beta > \mathfrak{n}_\alpha$ for $\beta > \alpha$ and

$$\mathfrak{n} = \mathfrak{n}_1 + \mathfrak{n}_2 + \cdots + \mathfrak{n}_\varkappa + \cdots.$$

Let $E_\varkappa$ be the set of elements x of E for which the cardinal number of the elements $y \in E$, for which xRy, is $< \mathfrak{n}_\varkappa$. Put

$$F_\varkappa = E_\varkappa - \sum_{\nu < \varkappa} E_\nu.$$

Clearly

$$E = \sum_\varkappa F_\varkappa.$$

As the theorem holds when $\mathfrak{n}$ is regular we obtain that $F_\varkappa$ may be decomposed into the sum of $\mathfrak{n}_\varkappa$ of fewer free subsets. As $\bar{\varkappa} < \mathfrak{n}$ and $\overline{\overline{\{F_\varkappa\}}} \leqq \mathfrak{n}$, for each $\varkappa$ it follows that E may be split off into the sum of $\mathfrak{n}$ or fewer free subsets. Thus Theorem 1 is proved in the general case too.

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