Rédeian skew product of operator groups.

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§ 1. Introduction.

One of the most important questions in group theory is the construction of new groups from given ones. The classical method of direct product[1]) has been generalized in several ways; let us mention only SCHREIER's extension theory[2]) or the Zappa—Szép extension of two groups[3]). Recently L. RÉDEI has introduced a fundamental method for constructing new groups from two given groups[4]). The new group has been called by him the *skew product* of the two factors. The great importance of the Rédeian skew product lies in the fact that it contains as special cases SCHREIER's and ZAPPA—SZÉP's extension theory as well as has several interesting applications to other problems of group theory.

In recent developments of abstract algebra, an important rôle is played by groups with operator domains. Therefore it seems to be desirable to extend RÉDEI's theory to groups with operators. The present paper is devoted to discussing this problem.

We start with two groups G and Γ with the same operator domain[5]) Ω and wish to get a survey over the Rédeian skew products[6]) $\mathfrak{G}=G\circ\Gamma$ of G and Γ such that $\mathfrak{G}$ can be made into an operator group with the same operator domain Ω. The effect of an operator $\in\Omega$ on the elements of $\mathfrak{G}$ may be defined so general that the treatment of the problem would be superfluously tedious and the problem itself would lose much of its interest. Therefore we shall confine ourselves to a particular case only. In selecting this case our leading viewpoint was that the skew product of operator groups shall contain

1) The group operation will be written as multiplication.

2) See SCHREIER [3] or ZASSENHAUS [6]. The numbers in brackets refer to the Bibliography given at the end of the paper.

3) See ZAPPA [5] and SZÉP [4].

4) RÉDEI [2].

5) This means not only that the elements of Ω act as operators both in G and Γ, but also that the product AB of two elements A, B of Ω has the same effect on the elements of both G and Γ as the successive application of A and B.

6) This notation is due to RÉDEI [2]

as special cases the important Schreier and Zappa—Szép extensions of operator groups, and besides, that the theory shall keep the symmetry in the two given groups. This intention leads us at once to the definition given in (10).

Our main result is Theorem 1 which gives a necessary and sufficient condition that a Rédeian skew product of two operator groups with the same operator domain shall be a group with the same domain of operators. Then we discuss those special cases of this skew product which correspond to SCHREIER's and ZAPPA—SZÉP's extension theory in case of operator groups. Splitting extensions are also considered.

§ 2. The main result.

Let G and Γ be two arbitrary groups with the common operator domain Ω. The elements of G will be denoted by italics, those of Γ by small Greek letters, while capitals such as $\mathsf{A}, \mathsf{B}, \dots$ are reserved for the operators in Ω. The elements into which $a \in G$ and $\alpha \in \Gamma$ are carried by an operator A will be written as a^{A} and α^{A}, respectively.

Assume that $\mathfrak{G} = G \circ \Gamma$ is a Rédeian skew product of the operator groups G and Γ, and $\mathfrak{G}$ has the same operator domain Ω. The elements of $\mathfrak{G}$ are all pairs (a, α) ($a \in G$ and $\alpha \in \Gamma$), and the multiplication rule in $\mathfrak{G}$ reads as follows: [6])

$$(a, \alpha)(b, \beta) = (a b^{\alpha} \beta^{\alpha}, a^{b} \alpha^{b} \beta) \tag{1}$$

where $b^{\alpha}, \beta^{\alpha} \in G$ and $a^{b}, \alpha^{b} \in \Gamma$. If, for a moment, we ignore the operator domain Ω, then $\mathfrak{G} = G \circ \Gamma$ is a pure Rédeian skew product of the groups G and Γ (without operators) and therefore, by RÉDEI's result [4]), $\mathfrak{G}$ is a group with the identity (e, ε) (e is the identity in G and ε is that in Γ) obeying (1) if and only if the following conditions are satisfied for all elements $a, b, c \in G$ and $\alpha, \beta, \gamma \in \Gamma$: [7])

$$
\begin{array}{lcc}
(2) & a^{\varepsilon} = a,\ e^{\alpha} = \varepsilon^{\alpha} = \alpha^{\varepsilon} = e; & \alpha^{e} = \alpha,\ \varepsilon^{a} = e^{a} = a^{e} = \varepsilon; \\
(3) & c^{a^{b}} = c; & \gamma^{\alpha^{\beta}} = \gamma; \\
(4) & \gamma^{a^{b}} = e; & c^{\alpha^{\beta}} = \varepsilon; \\
(5) & b^{\alpha} c^{\alpha^{b}} = (bc)^{\alpha} (b^{c})^{\alpha}; & \gamma^{a^{\beta}} \beta^{a} = (\beta^{\gamma})^{a} (\gamma\beta)^{a}; \\
(6) & \beta^{\alpha} c^{\alpha\beta} = (c^{\beta})^{\alpha} (\beta^{c})^{\alpha}; & \gamma^{ba} b^{a} = (b^{\gamma})^{a} (\gamma^{b})^{a}; \\
(7) & \beta^{\alpha} \gamma^{\alpha\beta} = (\gamma^{\beta})^{\alpha} (\beta\gamma)^{\alpha}; & c^{ba} b^{a} = (cb)^{a} (c^{b})^{a}; \\
(8) & (b^{c}\gamma)^{\alpha} = (b^{c})^{\alpha} \gamma^{\alpha}; & (c\beta^{\gamma})^{a} = c^{a} (\beta^{\gamma})^{a}; \\
(9) & \gamma^{\alpha^{c^{\beta}} \beta^{c}} = \gamma^{\alpha\beta}; & c^{b^{\gamma} a^{\gamma^{b}}} = c^{ba}.
\end{array}
$$

In what follows we suppose that (2)—(9) hold.

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[7]) All the formulae of this paper are to be understood to hold for all the elements of G and Γ as well as for all the operators in Ω which occur in the formula under consideration. In what follows we tacitly assume this convention.

In order to describe the effect of an operator A on an element (a, α) of $\mathfrak{G} = G \circ \Gamma$, we introduce two sets of elements $\{\mathrm{A}^{\alpha}\}$ and $\{\mathrm{A}^{a}\}$ of G and Γ, respectively, which are defined for all operators $\mathrm{A} \in \Omega$ and for all $\alpha \in \Gamma$ and $a \in G$. Let the defining equation of these A^{α}'s and A^{a}'s be the following:

$$(a, \alpha)^{\mathrm{A}} = (a^{\mathrm{A}} \mathrm{A}^{\alpha}, \mathrm{A}^{a} \alpha^{\mathrm{A}}) \tag{10}$$

where — we emphasize — A^{α} and A^{a} are uniquely determined elements of G and Γ, respectively. Our next aim is to establish the characteristic properties of the sets $\{\mathrm{A}^{\alpha}\}$ and $\{\mathrm{A}^{a}\}$.

First of all we observe that for all $\mathrm{A} \in \Omega$ we have

$$(e, \varepsilon)^{\mathrm{A}} = (e, \varepsilon),$$

considering that (e, ε) is the identity element of $\mathfrak{G} = G \circ \Gamma$. Therefore, by (10), we obtain

$$\mathrm{A}^{\varepsilon} = e, \quad \mathrm{A}^{e} = \varepsilon \qquad \text{for all } \mathrm{A} \in \Omega \tag{11}$$

(on account of $e^{\mathrm{A}} = e$ and $\varepsilon^{\mathrm{A}} = \varepsilon$).

Since each $\mathrm{A} \in \Omega$ is an operator acting on the elements of $\mathfrak{G}$, we must have

$$((a, \alpha)(b, \beta))^{\mathrm{A}} = (a, \alpha)^{\mathrm{A}} (b, \beta)^{\mathrm{A}} \tag{12}$$

for all (a, α) and (b, β) in $\mathfrak{G}$. The element on the left hand side of this equation is

$$((a, \alpha)(b, \beta))^{\mathrm{A}} = (a b^{\alpha} \beta^{\alpha}, a^{b} \alpha^{b} \beta)^{\mathrm{A}} = (a^{\mathrm{A}} (b^{\alpha})^{\mathrm{A}} (\beta^{\alpha})^{\mathrm{A}} \mathrm{A}^{a^{b} \alpha^{b} \beta}, \mathrm{A}^{a b^{\alpha} \beta^{\alpha}} (a^{b})^{\mathrm{A}} (\alpha^{b})^{\mathrm{A}} \beta^{\mathrm{A}}).$$

while that on the right hand side is

$$\begin{aligned}(a, \alpha)^{\mathrm{A}} (b, \beta)^{\mathrm{A}} &= (a^{\mathrm{A}} \mathrm{A}^{\alpha}, \mathrm{A}^{a} \alpha^{\mathrm{A}})(b^{\mathrm{A}} \mathrm{A}^{\beta}, \mathrm{A}^{b} \beta^{\mathrm{A}}) = \\ &= (a^{\mathrm{A}} \mathrm{A}^{\alpha} (b^{\mathrm{A}} \mathrm{A}^{\beta})^{\mathrm{A}^{a} \alpha^{\mathrm{A}}} (\mathrm{A}^{b} \beta^{\mathrm{A}})^{\mathrm{A}^{a} \alpha^{\mathrm{A}}}, (a^{\mathrm{A}} \mathrm{A}^{\alpha})^{b^{\mathrm{A}} \mathrm{A}^{\beta}} (\mathrm{A}^{a} \alpha^{\mathrm{A}})^{b^{\mathrm{A}} \mathrm{A}^{\beta}} \mathrm{A}^{b} \beta^{\mathrm{A}}).\end{aligned}$$

Therefore we have the following relations:

$$(b^{\alpha})^{\mathrm{A}} (\beta^{\alpha})^{\mathrm{A}} \mathrm{A}^{a^{b} \alpha^{b} \beta} = \mathrm{A}^{\alpha} (b^{\mathrm{A}} \mathrm{A}^{\beta})^{\mathrm{A}^{a} \alpha^{\mathrm{A}}} (\mathrm{A}^{b} \beta^{\mathrm{A}})^{\mathrm{A}^{a} \alpha^{\mathrm{A}}} \tag{13}$$

and

$$\mathrm{A}^{a b^{\alpha} \beta^{\alpha}} (a^{b})^{\mathrm{A}} (\alpha^{b})^{\mathrm{A}} = (a^{\mathrm{A}} \mathrm{A}^{\alpha})^{b^{\mathrm{A}} \mathrm{A}^{\beta}} (\mathrm{A}^{a} \alpha^{\mathrm{A}})^{b^{\mathrm{A}} \mathrm{A}^{\beta}} \mathrm{A}^{b} \tag{14}$$

(we have already cancelled a^{A} on the left in (13) and β^{A} on the right in (14)). We observe that (13) and (14) are dual to each other in the sense that one of them is obtained from the other by exchanging Latin and Greek letters and reversing the order of the factors.

Equations (13) and (14) are too complicated to work with them; therefore we shall break down them into several equations of simpler type. In view of the mentioned duality, it clearly suffices to consider only one of them, say (13).

We put $a = e$ in (13) and take into account that by (11) we have $\mathrm{A}^{e} = \varepsilon$ and by (2) $e^{b} = \varepsilon$. Thus we obtain

$$(b^{\alpha})^{\mathrm{A}} (\beta^{\alpha})^{\mathrm{A}} \mathrm{A}^{\alpha^{b} \beta} = \mathrm{A}^{\alpha} (b^{\mathrm{A}} \mathrm{A}^{\beta})^{\alpha^{\mathrm{A}}} (\mathrm{A}^{b} \beta^{\mathrm{A}})^{\alpha^{\mathrm{A}}}. \tag{15}$$

If we set here firstly $b=e$, secondly $\beta=\varepsilon$ and use (2), (11) again, we arrive at the following two relations:

(16) $$(\beta^{\alpha})^{A} A^{\alpha\beta} = A^{\alpha}(A^{\beta})^{\alpha A}(\beta^{A})^{\alpha A}$$

and

(17) $$(b^{\alpha})^{A} A^{\alpha^{b}} = A^{\alpha}(b^{A})^{\alpha A}(A^{b})^{\alpha A},$$

respectively.

Returning to (13), put there $\alpha=\varepsilon$ to get

(18) $$b^{A} A^{a^{b}\beta} = (b^{A} A^{\beta})^{A^{a}}(A^{b}\beta^{A})^{A^{a}}.$$

Choosing $b=e$, (18) implies

(19) $$A^{\beta} = (A^{\beta})^{A^{a}}(\beta^{A})^{A^{a}}.$$

Again from (18), with $\beta=\varepsilon$, it follows

(20) $$b^{A} A^{a^{b}} = (b^{A})^{A^{a}}(A^{b})^{A^{a}}.$$

Further conditions are obtained from the equation

(21) $$((a, \alpha)^{A})^{B} = (a, \alpha)^{AB},$$

where A and B are operators in Ω and AB is their product in Ω. Using (10), we find

$$((a, \alpha)^{A})^{B} = (a^{A} A^{\alpha}, A\ \alpha^{A})^{B} = (a^{AB}(A^{\alpha})^{B} B^{A^{a}\alpha A}, B^{a^{A}A^{\alpha}}(A^{a})^{B}\alpha^{AB})$$

and

$$(a, \alpha)^{AB} = (a^{AB}(AB)^{\alpha}, (AB)^{a}\alpha^{AB});$$

whence we have

(22) $$(A^{\alpha})^{B} B^{A^{a}\alpha A} = (AB)^{\alpha}$$

as well as

(23) $$B^{a^{A}A^{\alpha}}(A^{a})^{B} = (AB)^{a}.$$

(22) and (23) are again dual to each other, therefore we may consider only (22). Putting first $a=e$, then $\alpha=\varepsilon$, we are led to the relations

(24) $$(A^{\alpha})^{B} B^{\alpha A} = (AB)^{\alpha}$$

and

(25) $$B^{A^{a}} = e,$$

respectively.

Now we are going to formulate the main result of the present paper.

Theorem 1. *Let G and Γ be two operator groups with the common operator domain Ω. The set $\mathfrak{G} = G \circ \Gamma$ of all pairs (a, α) ($a \in G$ and $\alpha \in \Gamma$) under the rules*

(i) *equality: $(a, \alpha) = (b, \beta)$ is equivalent to $a = b, \alpha = \beta$,*
(ii) *multiplication: $(a, \alpha)(b, \beta) = (a\, b^{\alpha}\beta^{a}, a^{\beta}\alpha^{b}\beta)$,*
(iii) *effect of operators: $(a, \alpha)^{A} = (a^{A} A^{\alpha}, A^{a}\alpha^{A})$ if $A \in \Omega$,*

is an operator group with the identity (e, ε) *and the same operator domain* Ω *if and only if besides conditions* (2)—(9) *of* L. Rédei *the following ones are satisfied:*

(26) $\mathrm{A}^{\varepsilon} = e;$ $\qquad \mathrm{A}^{e} = \varepsilon;$

(27) $(\beta^{\alpha})^{\mathrm{A}} \mathrm{A}^{\alpha\beta} = \mathrm{A}^{\alpha} (\mathrm{A}^{\beta})^{\alpha^{\mathrm{A}}} (\beta^{\mathrm{A}})^{\alpha^{\mathrm{A}}};$ $\qquad \mathrm{A}^{ba} (b^{a})^{\mathrm{A}} = (b^{\mathrm{A}})^{a^{\mathrm{A}}} (\mathrm{A}^{b})^{a^{\mathrm{A}}} \mathrm{A}^{a};$

(28) $(b^{\alpha})^{\mathrm{A}} \mathrm{A}^{\alpha^{b}} = \mathrm{A}^{\alpha} (b^{\mathrm{A}})^{\alpha^{\mathrm{A}}} (\mathrm{A}^{b})^{\alpha^{\mathrm{A}}};$ $\qquad \mathrm{A}^{a^{\beta}} (\beta^{a})^{\mathrm{A}} = (\mathrm{A}^{\beta})^{a^{\mathrm{A}}} (\beta^{\mathrm{A}})^{a^{\mathrm{A}}} \mathrm{A}^{a};$

(29) $\mathrm{A}^{\beta} = (\mathrm{A}^{\beta})^{\mathrm{A}^{\alpha}} (\beta^{\mathrm{A}})^{\mathrm{A}^{\alpha}};$ $\qquad \mathrm{A}^{b} = (b^{\mathrm{A}})^{\mathrm{A}^{a}} (\mathrm{A}^{b})^{\mathrm{A}^{a}};$

(30) $b^{\mathrm{A}} \mathrm{A}^{\alpha^{b}} = (b^{\mathrm{A}})^{\mathrm{A}^{\alpha}} (\mathrm{A}^{b})^{\mathrm{A}^{\alpha}};$ $\qquad \mathrm{A}^{a^{\beta}} \beta^{\mathrm{A}} = (\mathrm{A}^{\beta})^{\mathrm{A}^{a}} (\beta^{\mathrm{A}})^{\mathrm{A}^{a}}:$

(31) $(\mathrm{A}^{\alpha})^{\mathrm{B}} \mathrm{B}^{\alpha^{\mathrm{A}}} = (\mathrm{A}\mathrm{B})^{\alpha};$ $\qquad \mathrm{B}^{a^{\mathrm{A}}} (\mathrm{A}^{a})^{\mathrm{B}} = (\mathrm{A}\mathrm{B})^{a};$

(32) $\mathrm{B}^{\mathrm{A}^{\alpha}} = e;$ $\qquad \mathrm{B}^{\mathrm{A}^{a}} = \varepsilon.$

Proof. We have to show that a given system $\mathfrak{G} = G \circ \Gamma$ with (i), (ii) and (iii) is a group with the identity (e, ε) and with the operator domain Ω if and only if all of (2)—(9) and (26)—(32) hold. Since conditions (2)—(9) are necessary and sufficient for $\mathfrak{G}$ being a group with the identity (e, ε) and with (i), (ii), it remains to be proved that (26)—(32) are necessary and sufficient conditions for the group $\mathfrak{G}$ to have the operator domain Ω and to obey the rule (iii).

The equations on the left sides of (26)—(32) were obtained above as necessary conditions; those on the right sides are their duals; consequently, the proof of the necessity is complete.

In order to prove the sufficiency, let us suppose that (26)—(32) hold. (26) implies that $(e, \varepsilon)^{\mathrm{A}} = (e, \varepsilon)$ for all $\mathrm{A} \in \Omega$. In proving (12), we proceed stepwise from (27)—(30), by making use of the associative law in $\mathfrak{G}$.

(27_1) and (30_2) imply

$$((e, \alpha)(e, \beta))^{\mathrm{A}} = (e, \alpha)^{\mathrm{A}} (e, \beta)^{\mathrm{A}}, \tag{33}$$

while (30_1) and (27_2) imply

$$((a, \varepsilon)(b, \varepsilon))^{\mathrm{A}} = (a, \varepsilon)^{\mathrm{A}} (b, \varepsilon)^{\mathrm{A}}. \tag{34}$$

Indeed, these are immediate consequences of the method by which we have arrived at (16) and (20).

For similar reasons we obtain from (28)

$$((e, \alpha)(a, \varepsilon))^{\mathrm{A}} = (e, \alpha)^{\mathrm{A}} (a, \varepsilon)^{\mathrm{A}}, \tag{35}$$

and from (29)

$$((a, \varepsilon)(e, \alpha))^{\mathrm{A}} = (a, \varepsilon)^{\mathrm{A}} (e, \alpha)^{\mathrm{A}}. \tag{36}$$

Next we prove the very special cases of (12):

$$((ab, a^{b})(e, \beta))^{\mathrm{A}} = (ab, a^{b})^{\mathrm{A}} (e, \beta)^{\mathrm{A}} \tag{37}$$

and

$$((b^{\alpha}, \alpha^{b})(e, \beta))^{\mathrm{A}} = (b^{\alpha}, \alpha^{b})^{\mathrm{A}} (e, \beta)^{\mathrm{A}}, \tag{38}$$

by making use of (33)—(36). Since

$$(ab, a^b)(e, \beta) = (ab\beta^{a^b}, a^b\beta) = (ab, \varepsilon)(e, a^b\beta)$$

(use (4)) and

$$(b^\alpha, \alpha^b)(e, \beta) = (b^\alpha\beta^{\alpha^b}, \alpha^b\beta) = (b^\alpha\beta^{\alpha^b}, \varepsilon)(e, \alpha^b\beta),$$

we have by (36), (33) and again (36)

$$((ab, a^b)(e, \beta))^{\mathsf{A}} = (ab, \varepsilon)^{\mathsf{A}}(e, a^b\beta)^{\mathsf{A}} = (ab, \varepsilon)^{\mathsf{A}}(e, a^b)^{\mathsf{A}}(e, \beta)^{\mathsf{A}} = (ab, a^b)^{\mathsf{A}}(e, \beta)^{\mathsf{A}},$$

and applying in turn (36), (34), (36), (33), (36), we obtain

$$\begin{aligned}((b^\alpha, \alpha^b)(e, \beta))^{\mathsf{A}} &= (b^\alpha\beta^{\alpha^b}, \varepsilon)^{\mathsf{A}}(e, \alpha^b\beta)^{\mathsf{A}} = (b^\alpha, \varepsilon)^{\mathsf{A}}(\beta^{\alpha^b}, \varepsilon)^{\mathsf{A}}(e, \alpha^b\beta)^{\mathsf{A}} = \\ &= (b^\alpha, \varepsilon)^{\mathsf{A}}(\beta^{\alpha^b}, \alpha^b\beta)^{\mathsf{A}} = (b^\alpha, \varepsilon)^{\mathsf{A}}(e, \alpha^b)^{\mathsf{A}}(e, \beta)^{\mathsf{A}} = (b^\alpha, \alpha^b)^{\mathsf{A}}(e, \beta)^{\mathsf{A}},\end{aligned}$$

which prove (37) and (38), respectively.

By (37), (34) and (36) we get

$$\begin{aligned}(39)\qquad ((a, \varepsilon)(b, \beta))^{\mathsf{A}} &= ((a, \varepsilon)(b, \varepsilon)(e, \beta))^{\mathsf{A}} = ((ab, a^b)(e, \beta))^{\mathsf{A}} = \\ &= (ab, a^b)^{\mathsf{A}}(e, \beta)^{\mathsf{A}} = (a, \varepsilon)^{\mathsf{A}}(b, \varepsilon)^{\mathsf{A}}(e, \beta)^{\mathsf{A}} = (a, \varepsilon)^{\mathsf{A}}(b, \beta)^{\mathsf{A}},\end{aligned}$$

and similarly, from (38), (35) and (36) we infer

$$\begin{aligned}(40)\qquad ((e, \alpha)(b, \beta))^{\mathsf{A}} &= ((e, \alpha)(b, \varepsilon)(e, \beta))^{\mathsf{A}} = ((b^\alpha, \alpha^b)(e, \beta))^{\mathsf{A}} = \\ &= (b^\alpha, \alpha^b)^{\mathsf{A}}(e, \beta)^{\mathsf{A}} = (e, \alpha)^{\mathsf{A}}(b, \varepsilon)^{\mathsf{A}}(e, \beta)^{\mathsf{A}} = (e, \alpha)^{\mathsf{A}}(b, \beta)^{\mathsf{A}}.\end{aligned}$$

Now we are in a position to verify (12):

$$\begin{aligned}((a, \alpha)(b, \beta))^{\mathsf{A}} &= ((a, \varepsilon)(e, \alpha)(b, \beta))^{\mathsf{A}} = (a, \varepsilon)^{\mathsf{A}}((e, \alpha)(b, \beta))^{\mathsf{A}} = \\ &= (a, \varepsilon)^{\mathsf{A}}(e, \alpha)^{\mathsf{A}}(b, \beta)^{\mathsf{A}} = (a, \alpha)^{\mathsf{A}}(b, \beta)^{\mathsf{A}},\end{aligned}$$

where we have applied (39), (40) and (36), successively. Hence A is actually an operator of $\mathfrak{G}$.

To complete the proof of the sufficiency of the condition in Theorem 1, we must still show that among the operators A, considered as operators acting on $\mathfrak{G}$, the same rule of composition holds as that originally given in Ω. In other words, (21) is to be verified. Considering that the validity of

$$((e, \alpha)^{\mathsf{A}})^{\mathsf{B}} = (e, \alpha)^{\mathsf{AB}} \quad \text{and} \quad ((a, \varepsilon)^{\mathsf{A}})^{\mathsf{B}} = (a, \varepsilon)^{\mathsf{AB}}$$

is an immediate consequence of (31_1), (32_2) and (32_1), (31_2), respectively, by (12), already proved, we conclude that

$$((a, \alpha)^{\mathsf{A}})^{\mathsf{B}} = ((a, \varepsilon)^{\mathsf{A}}(e, \alpha)^{\mathsf{A}})^{\mathsf{B}} = ((a, \varepsilon)^{\mathsf{A}})^{\mathsf{B}}((e, \alpha)^{\mathsf{A}})^{\mathsf{B}} = (a, \varepsilon)^{\mathsf{AB}}(e, \alpha)^{\mathsf{AB}} = (a, \alpha)^{\mathsf{AB}}.$$

This completes the proof of the theorem.

We observe that the existence of a Rédeian skew product of any two operator groups G and Γ is ensured by the fact that the direct product $G \times \Gamma$ of G and Γ always exists. Indeed, if we put

$$b^\alpha = b, \beta^\alpha = e, \quad a^b = \varepsilon, \alpha^b = \alpha, \quad \mathsf{A}^a = e, \mathsf{A}^\alpha = \varepsilon,$$

then all of (2)—(9) and (26)—(32) are automatically satisfied.

§ 3. Schreier extension of operator groups.

The Rédeian skew product of groups contains as a particular case SCHREIER's extension theory for groups. It belongs to the functions $b^a = b$, $\beta^a = e$ and has the multiplication rule

$$(a, \alpha)(b, \beta) = (ab, a^b \alpha^b \beta). \tag{41}$$

In this case the non-trivial ones of conditions (2)—(9) are the following:

$$\alpha^e = \alpha, \quad \varepsilon^a = e^a = a^e = \varepsilon; \tag{42}$$

$$\gamma^a \beta^a = (\gamma\beta)^a; \tag{43}$$

$$\gamma^{ba} b^a = b^a (\gamma^b)^a; \tag{44}$$

$$c^{ba} b^a = (cb)^a (c^b)^a, \tag{45}$$

which are obtained from (2_2), (5_2), (6_2) and (7_2), respectively.

The analogue of the Schreier extension for operator groups may be defined as follows. Let G and Γ be two operator groups possessing the same operator domain Ω. A group $\mathfrak{G}$ with the operator domain Ω is called a *Schreier extension of Γ by G*, if $\mathfrak{G}$ has an Ω-admissible normal subgroup Γ^* operator-isomorphic to Γ and the factor-group $\mathfrak{G}/\Gamma^*$ is operator-isomorphic to G. We shall show that the extension theory of operator groups may be subsumed under the Rédeian skew product of operator groups given in § 2, viz. it belongs to the case

$$b^a = b, \quad \beta^a = e, \quad \mathrm{A}^a = e. \tag{46}$$

In view of (46), most of the conditions (26)—(32) become identically true. One may readily check that only the following conditions have to be satisfied:

$$\mathrm{A}^e = \varepsilon; \tag{47}$$

$$\mathrm{A}^{ba}(b^a)^{\mathrm{A}} = (b^{\mathrm{A}})^{a^{\mathrm{A}}} (\mathrm{A}^b)^{a^{\mathrm{A}}} \mathrm{A}^a; \tag{48}$$

$$\mathrm{A}^a(\beta^a)^{\mathrm{A}} = (\beta^{\mathrm{A}})^{a^{\mathrm{A}}} \mathrm{A}^a; \tag{49}$$

$$\mathrm{B}^{a^{\mathrm{A}}}(\mathrm{A}^a)^{\mathrm{B}} = (\mathrm{AB})^a, \tag{50}$$

which are obtained from (26_2), (27_2), (28_2) and (31_2), respectively.

The following theorem holds.

Theorem 2. *If G and Γ are two operator groups with the same operator domain Ω, then a group $\mathfrak{G}$ with the same operator domain Ω is a Schreier extension of Γ by G if and only if it is isomorphic to a group of all pairs (a, α) $(a \in G, \alpha \in \Gamma)$ with the identity (e, ε) and subject to the rules*

(i) *equality: $(a, \alpha) = (b, \beta)$ if and only if $a = b$, $\alpha = \beta$,*

(ii) *multiplication: $(a, \alpha)(b, \beta) = (ab, a^b \alpha^b \beta)$,*

(iii) *effect of operators: $(a, \alpha)^{\mathrm{A}} = (a^{\mathrm{A}}, \mathrm{A}^a \alpha^{\mathrm{A}})$ (if $\mathrm{A} \in \Omega$),*

provided that all of the conditions (42)—(45) and (47)—(50) hold.

By virtue of Theorem 1 and in view of the fact that under (46), conditions (2)—(9) and (26)—(32) are equivalent to (42)—(45) and (47)—(50), respectively, what we have to prove is that the set [8]) $\Gamma^* = (e, \Gamma)$ is an admissible subgroup of $\mathfrak{G}$, further that the isomorphism $(e, \alpha) \longleftrightarrow \alpha$ between Γ^* and Γ as well as the isomorphism $(a, \alpha) \longleftrightarrow a$ between the factor-group $\mathfrak{G}/\Gamma^*$ and G are operator-preserving. These statements may easily be checked. In fact, by (iii) and (47) we have

$$(e, \alpha)^{\mathrm{A}} = (e^{\mathrm{A}}, {}_{\mathrm{A}}\, \alpha^{\mathrm{A}}) = (e, \alpha^{\mathrm{A}}) \in \Gamma^*,$$

further

$$(e, \alpha)^{\mathrm{A}} = (e, \alpha^{\mathrm{A}}) \longleftrightarrow \alpha^{\mathrm{A}}$$

under the first indicated isomorphism and

$$(a, \alpha)^{\mathrm{A}} = (a^{\mathrm{A}}, \mathrm{A}''\alpha^{\mathrm{A}}) \longleftrightarrow a^{\mathrm{A}}$$

under the second one. Q. e. d.

§ 4. Zappa—Szép extension of operator groups.

If a group $\mathfrak{G}$ has two subgroups G^* and Γ^* such that each element of $\mathfrak{G}$ may be represented in a unique form $a^*\alpha^*$ with $a^* \in G^*$ and $\alpha^* \in \Gamma^*$, then $\mathfrak{G} = G^*\Gamma^*$ is called a *factorization* of $\mathfrak{G}$, and $\mathfrak{G}$ is said to be a *Zappa—Szép product* [9]) of the groups G^* and Γ^*. The same definition may be applied to operator groups with the sole modification that both G^* and Γ^* are assumed to be admissible subgroups of $\mathfrak{G}$.

Conversely, if given two operator groups G and Γ with the same operator domain Ω and $\mathfrak{G}$ is an operator group with the same Ω such that $\mathfrak{G} = G^*\Gamma^*$ is a factorization of $\mathfrak{G}$ where the admissible subgroups G^* and Γ^* of $\mathfrak{G}$ are operator-isomorphic to G and Γ, respectively, then $\mathfrak{G}$ is called *a Zappa—Szép extension of the operator groups G and Γ*. This extension is a particular case of the Rédeian skew product of operator groups. In fact, the following theorem holds.

Theorem 3. *Let G and Γ be two operator groups with the same operator domain Ω. A group $\mathfrak{G}$ with the same operator domain Ω is a Zappa—Szép extension of G and Γ if and only if it is isomorphic to a group of all pairs (a, α) ($a \in G$ and $\alpha \in \Gamma$) with the identity (e, ε) and with the rules*

(i) *equality:* $(a, \alpha) = (b, \beta)$ *is equivalent to* $a = b, \alpha = \beta$,

(ii) *multiplication:* $(a, \alpha)(b, \beta) = (ab^{\alpha}, \alpha^{b}\beta)$,

(iii) *effect of operators:* $(a, \alpha)^{\mathrm{A}} = (a^{\mathrm{A}}, \alpha^{\mathrm{A}})$ (*if* $\mathrm{A} \in \Omega$),

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[8]) If D and Δ are any subsets of G and Γ, respectively, then (D, Δ) denotes the set of all (a, α) of $\mathfrak{G}$ such that $a \in D$ and $\alpha \in \Delta$.

[9]) Cf. Zappa [5] and Szép [4].

such that the following conditions are satisfied:

$$a^{\varepsilon}=a, \quad e^{\alpha}=e; \qquad \alpha^{a}=\alpha, \quad \varepsilon^{a}=\varepsilon; \tag{51}$$

$$b^{\alpha}c^{\alpha^{b}}=(bc)^{\alpha}; \qquad \gamma^{a^{\beta}}\beta^{a}=(\gamma\beta)^{a}; \tag{52}$$

$$c^{\alpha\beta}=(c^{\beta})^{\alpha}; \qquad \gamma^{ba}=(\gamma^{b})^{a}; \tag{53}$$

$$(b^{\alpha})^{\mathsf{A}}=(b^{\mathsf{A}})^{\alpha^{\mathsf{A}}}; \qquad (\beta^{a})^{\mathsf{A}}=(\beta^{\mathsf{A}})^{a^{\mathsf{A}}}. \tag{54}$$

P r o o f. Applying Theorem 1 with

$$\alpha^{\beta}=e, \quad a^{b}=\varepsilon, \quad \mathsf{A}^{\alpha}=e, \quad \mathsf{A}^{a}=\varepsilon, \tag{55}$$

we see that the Rédeian skew product of the operator groups G and Γ exists, in view of the fact that conditions (2)—(9) and (26)—(32) are reduced by (55) to trivial ones and to (51)—(54). (The non-trivial ones enumerated under (51)—(54) are obtained from (2), (5), (6) and (28), respectively.) Therefore, it remains to be proved that $G^{*}=(G,\varepsilon)$ and $\Gamma^{*}=(e,\Gamma)$ are admissible subgroups of $\mathfrak{G}$ and the one-to-one mappings

$$(a,\varepsilon)\longleftrightarrow a \quad \text{and} \quad (e,\alpha)\longleftrightarrow\alpha \tag{56}$$

between G^{*} and G, resp. between Γ^{*} and Γ are operator-isomorphisms. That G^{*} and Γ^{*} are subgroups of $\mathfrak{G}$ and are isomorphic to G and Γ, respectively, under the isomorphisms (56), is well known [4]). That G^{*} and Γ^{*} are admissible subgroups follows at once from (iii):

$$(a,\varepsilon)^{\mathsf{A}}=(a^{\mathsf{A}},\varepsilon) \quad \text{and} \quad (e,\alpha)^{\mathsf{A}}=(e,\alpha^{\mathsf{A}}).$$

This also shows that the isomorphisms (56) are operator-preserving, as we wished to prove.

§ 5. Splitting extensions.

There is a particular case of the Rédeian skew product which is a common subcase of SCHREIER's and ZAPPA—SZÉP's extension theory. This is the so-called *splitting extension* and corresponds to the functions

$$b^{\alpha}=b, \quad a^{b}=\varepsilon, \quad \beta^{\alpha}=e. \tag{57}$$

In case of operator groups we add to these the conditions

$$\mathsf{A}^{\alpha}=e, \quad \mathsf{A}^{a}=\varepsilon. \tag{58}$$

Under (57) and (58), conditions (2)—(9) and (26)—(32) reduce to the following relations (the trivial ones are omitted):

$$\alpha^{a}=\alpha; \quad \varepsilon^{a}=\varepsilon; \tag{59}$$

$$\gamma^{a}\beta^{a}=(\gamma\beta)^{a}; \tag{60}$$

$$\gamma^{ba}=(\gamma^{b})^{a}; \tag{61}$$

$$(\beta^{a})^{\mathsf{A}}=(\beta^{\mathsf{A}})^{a^{\mathsf{A}}}, \tag{62}$$

which are obtained from (2_2), (5_2), (6_2) and (28_2), respectively.

From Theorem 1, or, equivalently, from Theorem 2 or 3, we conclude:

Theorem 4. *If G and Γ are two operator groups with the same operator domain Ω, then the set of all pairs (a, α) $(a \in G, \alpha \in \Gamma)$ subject to the rules*

(i) *equality:* $(a, \alpha) = (b, \beta)$ *is equivalent to* $a = b, \alpha = \beta$,

(ii) *multiplication:* $(a, \alpha)(b, \beta) = (ab, \alpha^b \beta)$,

(iii) *effect of operators:* $(a, \alpha)^{\mathrm{A}} = (a^{\mathrm{A}}, \alpha^{\mathrm{A}})$ $(\mathrm{A} \in \Omega)$

is a group with the identity (e, ε) and with the same Ω if and only if (59) *to* (62) *are satisfied.*

In the splitting case the subsets $G^* = (G, \varepsilon)$ and $\Gamma^* = (e, \Gamma)$ of $\mathfrak{G}$ are admissible subgroups, moreover Γ^* is a normal subgroup of $\mathfrak{G}$. Among all Schreier extensions of the operator groups G and Γ the splitting ones are distinguished by the property that the elements of the representation system $\{(a, \varepsilon)\}$ for $\mathfrak{G}$ modulo Γ^* constitute a group (Ω-isomorphic to G).

More generally, we shall say that a Schreier extension of Γ by G *splits over* Γ if $\mathfrak{G}$ has a representation system[10] $\{(a, \alpha_a)\}$ whose elements form an admissible subgroup of $\mathfrak{G}$. A group $\mathfrak{H}$ with the same operator domain Ω is said to be *a splitting group for $\mathfrak{G}$ modulo Γ relative to the representation system* $\{(a, \varepsilon)\}$ if (I) $\mathfrak{G}$ may be imbedded Ω-isomorphically in $\mathfrak{H}$, (II) $\mathfrak{H}$ has an admissible normal subgroup H containing Γ^*, (III) $\{(a, \varepsilon)\}$ is a representation system for $\mathfrak{H}$ modulo H and (IV) $\mathfrak{H}$ splits over H. We shall show that *such a splitting extension always exists.*

In fact[11]), consider the direct product $\mathfrak{G} \times G = \mathfrak{H}$, formed by the elements $(a, \alpha)b$ $((a, \alpha) \in \mathfrak{G}, b \in G)$. For convenience we identify the element $(a, \alpha)e$ with (a, α) of $\mathfrak{G}$ and the element $(e, \varepsilon)b$ with b of G. By this construction, (I) is trivially satisfied. The elements of $\mathfrak{H}$ which have the form $(a, \alpha)a$ constitute an admissible subgroup H of $\mathfrak{H}$ which is Ω-isomorphic to G under the correspondence $(a, \alpha)a \longleftrightarrow (a, \alpha)$. This H is clearly a normal subgroup of $\mathfrak{H}$ and contains the set of all $(e, \alpha)e = (e, \alpha)$, that is, it contains Γ^*. Hence (II) holds. Condition (III) is also satisfied, since $\{(a, \varepsilon)\}$ is obviously a complete representation system for $\mathfrak{H}$ modulo H. Finally, $\mathfrak{H}$ splits over H, for the elements $(e, \varepsilon)a = a$ constitute a complete representation system for $\mathfrak{H}$ modulo H and form an admissible subgroup of $\mathfrak{H}$. This establishes our statement.

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[10]) For each $a \in G$, there is exactly one $\alpha_a \in \Gamma$ such that (a, α_a) belongs to this representation system.

[11]) The following method is due to Everett [1].

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(Received June 5, 1952.)