On the structure of semi-modular lattices of infinite length.

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1. Introduction. In the applications of lattice theory to modern algebra and geometry we have to do mostly with lattices in which the so-called Jordan—Dedekind chain condition holds:

(JD) a, b being arbitrary elements of the lattice such that $a<b$, all maximal chains[1]) between a and b have the same length.

Therefore, in view of these applications, it is of some importance to have conditions which imply (JD).

It is well-known[2]) that for the fulfilment of (JD) it is sufficient that the elements of the lattice satisfy the following two conditions:

(α) If x covers[3]) $x \cap y$, then $x \cup y$ covers y;

(β) If a, b are arbitrary elements of the lattice such that $a<b$, then every chain between a and b is finite.

In the first part of this paper we show that condition (β) may be replaced by a weaker condition which we shall formulate in the corollary to theorem 1. Using this theorem, we shall establish then some facts concerning the structure of a special family of lattices.

It is known that in the case of lattices of finite length the concept of semi-modularity is usually defined just by condition (α); moreover, that in this case (α) is eqivalent to each of the following conditions:

(γ) If x and y cover a and $x \neq y$, then $x \cup y$ covers x and y;

(δ) $x \cap y < z < x < x \cup y$ implies that there exists an element t such that $(x \cap y) < t \leqq y$ and $x \cap (t \cup z) = z$.

For lattices of infinite length, this equivalence does not remain valid in

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1) A chain $x_0 < x_1 < \ldots < x_r$ is called maximal (and of length r) if there is no element t such that $x_{i-1} < t < x_i$ for an i $(1 \leqq i \leqq r)$. See, for the usual symbols and terminology, G. BIRKHOFF, *Lattice theory*, revised edition (New York, 1948).

2) G. BIRKHOFF, op. cit. pp. 66—67.

3) "a covers b" means that $a > b$ and that there is no element t such that $a > t > b$.

general[4])[5]) and, in this case, there is no generally adopted definition of semi-modularity. We shall use, following R. CROISOT[6]), the

Definition: *A lattice (of finite or infinite length) is called semi-modular if and only if it satisfies condition* (δ).

Condition (δ) implies[5]) condition (α); thus, in the second and third parts of our paper — where we shall treat only semi-modular lattices — we can make use of (α). Often it is useful to write (α) in the following form:

(α') If x covers a and $y \geqq a$, then either $x \cup y = y$ or $x \cup y$ covers y.

2. A theorem on maximal chains. For "x covers y" we shall use the symbol $x \succ y$; by $x \succeq y$ we shall denote that either $x \succ y$ or $x = y$.

First we shall prove the following

Theorem 1. *Let L be a lattice satisfying* (α), *and let a, b be elements of L such that $a < b$. If there exists a maximal chain*

$$a = a_0 \prec a_1 \prec \cdots \prec a_r = b \tag{1}$$

of length r between a and b, then (i) *the length of any other chain between a and b is at most r;* (ii) *maximal chains between a and b are precisely of length r.*

Corollary. *Let L be a lattice satisfying* (α). *If there exists a finite maximal chain between two arbitrary elements a, b of L such that $a < b$, then condition* (JD) *holds in L.*

This means that for the fulfilment of (JD) it is sufficient to assume, instead of (β), merely the existence of a single finite maximal chain between every a and b ($a < b$).

Proof. It is sufficient to prove only theorem 1; the corollary follows by what has been said in the introduction. Further, it suffices to show only assertion (i) of the theorem. For, let

$$a = x_0 < x_1 < \cdots < b \tag{2}$$

be an arbitrary (i. e. not necessarily maximal) chain between a and b. If we denote the length of (2) by r', then by (i) we have $r' \leqq r$. But, by (i), the chain (2) may obviously be made maximal, of length r'_m say, and we have also $r'_m \leqq r$. Conversely, starting with this maximal chain of length r'_m, we can infer also $r \leqq r'_m$; thus for maximal chains we have $r'_m = r$, indeed.

Now, we prove our theorem by showing that (2) has no subchain of length $r+1$.

[4]) G. BIRKHOFF, op. cit. p. 102, ex. 4 (a).

[5]) R. CROISOT, Contribution à l'étude des treillis semi-modulaires de longueur infinie, *Annales de l'École Normale Supérieure*, 68 (1951), pp. 203—265; especially pp. 211, 215—216.

[6]) R. CROISOT, op. cit. p. 204.

If $r=1$, i. e. if

$$(1') \qquad a \prec b,$$

then our assertion is obvious. For $r \geqq 2$ we prove our assertion by complete induction with respect to the length of the finite maximal chain of the form (1).

Suppose that — contrary to our assertion — it is possible to choose from (2) a subchain

$$(3) \qquad a < y_1 < y_2 < \cdots < y_{r+1} = b$$

of length $r+1$. Consider now the joins of the elements of (3) taken with the element a_1 of (1). Obviously

$$(4) \qquad a_1 \leqq y_1 \cup a_1 \leqq y_2 \cup a_1 \leqq \cdots \leqq y_r \cup a_1 \leqq b,$$

i. e. all distinct elements of (4) form a chain between a_1 and b. Now, consider the subchain of (1) between the same elements:

$$(1^*) \qquad a_1 \prec a_2 \prec \cdots \prec a_r = b.$$

The length of (1^*) is $r-1$. Therefore, by the induction hypothesis, every chain between a_1 and b is of length $\leqq r-1$.

However, we shall show that under our assumption expressed in (3) the chain of all distinct elements in (4) is of length $\geqq r$. It will follow from this contradiction that our above assumption cannot be true: the elements of (3) may not be all distinct.

In order to show this fact, we distinguish three cases, according as

1°. $y_i \cup a_1 = y_i$ *for* $i = 1, 2, \ldots, r$,
2°. $y_i \cup a_1 \neq y_i$ *for* $i = 1, 2, \ldots, r$,
3°. $y_1 \cup a_1 \neq y_1$, *but* $y_i \cup a_1 = y_i$ *for some* $i, 1 < i \leqq r$.

This classification contains all possible cases, because $y_k \cup a_1 = y_k$ (for some k) is equivalent to $y_k \geqq a_1$; hence and from (3) $y_i \geqq a_1$, i. e. $y_i \cup a_1 = y_i$ also for all $i \geqq k$.

Case 1°. (4) gives in this case, by (3), the following chain between a_1 and b:

$$(4^*) \qquad a_1 \leqq y_1 < y_2 < \cdots < y_r < y_{r+1} = b.$$

But (4^*) is obviously at least of length r.

Case 2°. Then in particular $y_1 \cup a_1 \neq y_1$. This means that

$$(5) \qquad y_1 \cup a_1 > a_1.$$

For, $y_1 \cup a_1 = a_1$ would imply $y_1 \leqq a_1$; but $y_1 = a_1$ is impossible by our assumption $y_1 \cup a_1 \neq y_1$, while $a_1 > y_1$ is impossible because $a_1 \succ a$ and $y_1 > a$.

Further, since $a_1 \succ a$ and $y_i > a$, but $y_i \cup a_1 \neq y_i$ for all $i \leqq r$, (α') implies

$$y_i \cup a_1 \succ y_i \quad (i = 1, 2, \ldots, r).$$

By this fact and by (4) we have

(6′) $$y_{i+1}\cup a_1 \geqq y_i \cup a_1 \succ y_i$$

and

(6″) $$y_{i+1}\cup a_1 \succ y_{i+1} > y_i$$

for all $i \leqq r-1$. From (6′) and (6″) $y_{i+1}\cup a_1 \neq y_i \cup a_1$, so that

(7) $$y_{i+1}\cup a_1 > y_i \cup a_1$$

for $i=1,2,\ldots,r-1$. From (5) and (7) we have

(8) $$a_1 < y_1 \cup a_1 < y_2 \cup a_1 < \cdots < y_r \cup a_1 \leqq b.$$

Now, (8) is a chain between a_1 and b, whose length is at least r.

Case 3°. Let us denote, in this case, the least index $i>1$ for which $y_i \cup a_1 = y_i$, by l. Then, as we have established above, $y_i \cup a_1 = y_i$ hold also for all $i \geqq l$.

As $y_1 \cup a_1 \neq y_1$, (5) holds also in this case. Further, one can see by the same arguments as in case 2°, that (7) holds for all $i \leqq l-2$. For $i=l-1$ we cannot in general strengthen the sign $\leqq$ to $<$ in (4). For $i=l,\ l+1,\ldots,r,$ $r+1$, since we have $y_i \cup a_1 = y_i$, (7) follows immediately from (3). Thus we have in this case

(8′) $$a_1 < y_1 \cup a_1 < \cdots < y_{l-2}\cup a_1 < y_{l-1}\cup a_1 \leqq y_l \cup a_1 < \\ < y_{l+1}\cup a_1 < \cdots < y_r \cup a_1 < y_{r+1}\cup a_1 (=b),$$

i. e. we have found a chain between a_1 and b, which is again at least of length r.

3. Semi-modular semi-complemented lattices. A lattice with a least element O will be called *semi-complemented*, if for any element a (not equal to the eventually existing greatest element I of L) the equation $a \cap x = O$ has at least one solution $x \neq O$ in L; the element x will then be called a *semi-complement* of a.

Clearly, all complemented lattices are a fortiori semi-complemented.

On the other hand it is easy to give examples showing that semi-complementedness and existence of a greatest element do not imply complementedness even in the case of lattices of finite length.

It was shown in a previous paper[7]) that if one assumes also semi-modularity, then, for lattices of finite length, semi-complementedness implies complementedness. Moreover this equivalence holds also for lattices with greatest element and of infinite length provided that they satisfy also the infinite distributive laws. But, as the following simple counter-example shows, for lattices of infinite length, semi-complementedness does not imply necessarily complementedness even if one considers only distributive lattices with greatest element.

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[7]) G. Szász, Dense and semi-complemented lattices. To be published in *Nieuw Archief voor Wiskunde.*

Let L be the set of all finite subsets of a given countable set S, including also the void set. Let us define in L the ordering relation by the set-theoretical inclusion.. L is then a semi-complemented distributive lattice of infinite length; but it has no greatest element. However, by adjoining to L the whole set S as elements I, we can make it into a lattice L^* with a greatest and a least element. L^* is obviously semi-complemented and distributive, but not complemented.

The particularity of L^* lies in the property that, although it is of infinite length, yet all chains between an arbitrary element $a \neq I$ and the element O are finite. Such elements, which will play an important role in this section, will be called *of finite heigth.* We shall show that the behaviour of L^* is typical in the sense that if any lattice possesses the enumerated properties of L^*, then its elements have no complements. More precisely:

Theorem 2. *Let L be a semi-modular lattice of infinite length, having a greatest element I and a least element O. If, for each element $a \neq I$, there exists a finite maximal chain between O and a, then no element $\neq O, I$ has complements in L.*

Proof. Let a be an arbitrary element of L, $a \neq I$. If a has no semi-complement, then our statement is already proved for a. Thus we can assume that a has (at least) one semi-complement x.

By hypothesis there exists a finite maximal chain between O and a, and similarly between O and x. Thus, by theorem 1, a and x are both of finite height. For proving our theorem it is sufficient to find a finite maximal chain between a and $a \cup x$, for arbitrary a and x. In fact, let us consider a finite maximal chain between O and a, and continuate it by the finite maximal chain found between a and $a \cup x$. So we get a finite maximal chain between O and $a \cup x$. Hence, L being of infinite length and semi-modular, we can infer that $a \cup x \neq I$.

We begin therefore to construct a finite maximal chain between a and $a \cup x$. As we have already shown, all chains between O and x are finite. Let

$$O = x_0 \prec x_1 \prec \cdots \prec x_{r-1} \prec x_r = x \tag{9}$$

be a maximal chain. Consider the joins of these elements taken with a. Then we get the following series:

$$(a =)\, a \cup x_0 \leqq a \cup x_1 \leqq \cdots \leqq a \cup x_{r-1} \leqq a \cup x. \tag{10}$$

Now, for each value of i $(i = 0, 1, \ldots, r-1)$, there are two cases to distinguish according as $a \cup x_i \geqq x_{i+1}$ holds or not.

If $a \cup x_i \geqq x_{i+1}$, then we have also $a \cup x_i \geqq a \cup x_{i+1}$. Hence and from (10) we get

$$a \cup x_i = a \cup x_{i+1}. \tag{11}$$

If $a \cup x_i$ is not $\geqq x_{i+1}$, then $(a \cup x_i) \cap x_{i+1} < x_{i+1}$. But then $a \cup x_i \geqq x_i$ and $x_{i+1} \geqq x_i$ imply $x_{i+1} > (a \cup x_i) \cap x_{i+1} \geqq x_i$. Hence and from $x_{i+1} \succ x_i$ we conclude $(a \cup x_i) \cap x_{i+1} = x_i$. This means that

$$x_{i+1} \succ (a \cup x_i) \cap x_{i+1}. \tag{12}$$

Since the lattice is semi-modular, we have by (α) and from (12)

$$(a \cup x_i) \cup x_{i+1} \succ a \cup x_i. \tag{13}$$

Hence, by $(a \cup x_i) \cup x_{i+1} = a \cup (x_i \cup x_{i+1}) = a \cup x_{i+1}$, we get

$$a \cup x_{i+1} \succ a \cup x_i. \tag{14}$$

Thus, we have by (11) and (14)

$$(a =) a \cup x_0 \preceq a \cup x_1 \preceq \cdots \preceq a \cup x_{r-1} \preceq a \cup x,$$

i. e. all distinct elements of (10) form a finite maximal chain between a and $a \cup x$. Thus, by what has been told above, our theorem is proved.

This proof yields the following statement: if a is an element of finite height of the lattice L (satisfying the conditions of theorem 2) and if it has only semi-complements x with the same property, then none of the semi-complements x is a complement of a (i. e. a has no complement).

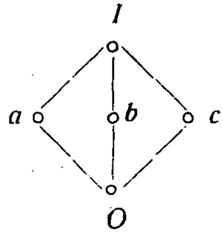
From this fact we get an interesting result for semi-complemented lattices L' of infinite length and with greatest element in which the infinite distributive laws hold. Indeed, as I have proved in my above-mentioned paper, such lattices L' are all complemented. Suppose now that L' has some element $a \neq I$ of finite height. (By theorem 1, for this property it is sufficient that a finite maximal chain exist between O an a.) Then it follows by the above fact that a has at least one semi-complement which is not of finite height. By theorem 1 we can formulate this result also in the following form:

Theorem 3. *Let L be a semi-complemented lattice of infinite length and with a greatest element I (and a least element O) in which the infinite distributive laws hold. If for some element $a \neq I$ there exists a finite maximal chain between O and a, then one can find a semi-complement x of a such that all maximal chains between O and x (if any) are of infinite length.*

4. On the number of points in semi-modular lattices. An element p of a lattice with O is called a *point* if $p \succ O$. In the lattice-theoretical treatment of Boolean algebras (i. e. complemented distributive lattices) of finite length it is a remarkable result that the length of such lattices is equal to the number of their points[8]). As all semi-complemented distributive lattices of finite length is also complemented, this result can be formulated also for *semi-complemented* distributive lattices of finite length.

[8]) This follows immediately from theorem 6 in G. Birkhoff, op. cit. p. 159.

But, considering the following non-distributive (but modular) lattice:



one can see immediately that the above statement holds in general only for distributive lattices. The question arises: what can be said about the length and the number of points in the case of semi-complemented modular or semi-modular lattices? In this direction we have the following

Theorem 4. *Let L be a semi-modular and semi-complemented lattice with greatest element I, in which to each element $a \neq O$ one can find at least one point p with $a \geqq p$. If L has only a finite number r of points, then its length is less than or equal to r.*

Proof. Let us denote the points of L by $p_1, p_2, \ldots, p_r$ (r finite), and a semi-complement of the element a by x. By assumption, $x \geqq p_k$ for some $k (\leqq r)$. Then $a \cap x = O$ implies $a \cap p_k = O$. Since L is semi-complemented, this means that to each a there is at least one p_i $(i \leqq r)$ for which $a \cap p_i = O$. Thus L cannot have elements greater than $\bigcup_{i=1}^{r} p_i$; but, since L has also the greatest element I, we have $\bigcup_{i=1}^{r} p_i = I$.

Consider now the sequence

$$(15) \qquad \begin{cases} x_0 = O \\ x_1 = x_0 \cup p_1 = p_1 \\ x_2 = x_1 \cup p_2 = p_1 \cup p_2 \\ \quad \vdots \\ x_k = x_{k-1} \cup p_k = p_1 \cup p_2 \cup \cdots \cup p_k \\ \quad \vdots \\ x_r = x_{r-1} \cup p_r = p_1 \cup p_2 \cup \cdots \cup p_r = I. \end{cases}$$

Obviously

$$O = x_0 \leqq x_1 \leqq x_2 \leqq \cdots \leqq x_r = I.$$

By theorem 1 it is sufficient to show that $x_k \succeq x_{k-1}$. Now we have $x_1 \succ x_0$ by definition. Let $k \geqq 2$. If $x_{k-1} \geqq p_k$, then $x_k = x_{k-1}$. If x_{k-1} is not $\geqq p_k$, then $p_k \succ O$ and $x_{k-1} \geqq O$ imply, by (α')

$$x_{k-1} \cup p_k \succ x_{k-1}, \quad \text{i. e.} \quad x_k \succ x_{k-1},$$

thus completing the proof of the theorem.

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