Approximation properties of orthogonal expansions.

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1. In the approximation theory of Fourier series one considers usually problems of the following type. Given a certain class C of periodic functions $f(x)$, and a non-negative finite valued functional $N(f)$ on C, one asks how rapidly the nth partial sum s_n (or the nth Fejér sum, etc.) of the Fourier series of f tends to f when $n \to \infty$, the distance of f and s_n being measured in the L^p sense with some $p \geqq 1$, the uniform approximation corresponding to $p = \infty$. More precisely, one considers the quantities

$$\varrho_n = \underset{f \in C}{\text{l. u. b.}} \frac{\|f - s_n\|_p}{N(f)}$$

and seeks appropriate estimates *from above* for ϱ_n when $n \to \infty$.

Classes particularly concerned are the Lipschitz classes, the classes of functions having a derivative belonging to a Lipschitz class, etc., the functional $N(f)$ being defined correspondingly as the least Lipschitz constant in question, etc.

W. RUDIN considered in a recent paper[1]), the problem of estimating the quantities ϱ_n *from below*, and not only for the Fourier series, but for *all orthogonal expansions*. He considered namely the approximation in the L^2 sense for the Lipschitz class Lip 1 and for the class C_V of functions of bounded variation. He showed that there are positive absolute constants A, B such that, for the partial sums of any orthogonal expansion on $[0, 1]$, $\varrho_n \geqq A n^{-1}$ for the class Lip 1, and $\varrho_n \geqq B n^{-\frac{1}{2}}$ for the class C_V. Since for the Fourier series expansion we have

$$\varrho_n = O(n^{-1}) \quad \text{and} \quad \varrho_n = O(n^{-\frac{1}{2}}),$$

respectively, it turns thus out that, for the two classes considered and for the approximation in L^2, there is no orthogonal system which would yield expansions converging faster than the Fourier series expansion.

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[1]) W. RUDIN, L^2-approximation by partial sums of orthogonal developments, *Duke Math. Journal*, 19 (1952), 1—4.

RUDIN's method depends in its original form rather strongly on properties of the space L^2. However, we shall show that it is possible to find a related method which applies also to approximations in the L^p sense, $p \geqq 1$. So we are able to derive estimates from below for the approximation in the L^p sense for various classes of functions, and for all orthonormal systems.

2. We need the following

Lemma. *Let $\varphi_1, \ldots, \varphi_n$ and $\psi_1, \ldots, \psi_m$ be two orthonormal sets of functions defined on the interval* $[0, 1]$ *i. e.*

$$\int_0^1 \varphi_i(x)\overline{\varphi_j(x)}\,dx = \delta_{ij}, \qquad \int_0^1 \psi_h(x)\overline{\psi_k(x)}\,dx = \delta_{hk},$$

and let $\lambda_1, \ldots, \lambda_n$ and $M_1, \ldots, M_m$ be given numbers such that

$$\sum_{i=1}^{n} |\lambda_i|^2 \leqq n, \qquad M_k > 0 \qquad (k = 1, \ldots, m)$$

For any $f \in L^2[0, 1]$ put

$$s_n(f) = s_n(f; x) = \sum_{i=1}^{n} \lambda_i c_i \varphi_i(x)$$

where

$$c_i = (f, \varphi_i) = \int_0^1 f(x)\overline{\varphi_i(x)}\,dx,$$

and consider, for $1 \leqq p \leqq 2$, the quantity

$$\alpha_p = \max_{1 \leqq k \leqq m} \frac{\|\psi_k - s_n(\psi_k)\|_p}{M_k},$$

where

(1) $$\|f\|_p = \left[\int_0^1 |f(x)|^p\,dx\right]^{\frac{1}{p}}.$$

Then

(2) $$\alpha_p \geqq \frac{\sum_{k=1}^{m} \|\psi_k\|_p - (mn)^{\frac{1}{2}}}{\sum_{k=1}^{m} M_k}$$

Proof. Introducing the kernel function

$$K(x, y) = \sum_{i=1}^{n} \lambda_i \varphi_i(x)\overline{\varphi_i(y)}$$

we can write

$$s_n(\psi_k) = \int_0^1 K(x, y)\psi_k(y)\,dy.$$

Using HÖLDER's inequality we get

$$\sum_{k=1}^{m} \|s_n(\psi_k)\|_p \leqq m^{\frac{1}{q}} \left\{ \sum_{k=1}^{m} \|s_n(\psi_k)\|_p^p \right\}^{\frac{1}{p}} =$$

$$= m \left\{ \frac{1}{m} \sum_{k=1}^{m} \int_0^1 \left| \int_0^1 K(x,y)\psi_k(y)\,dy \right|^p dx \right\}^{\frac{1}{p}} = m \cdot \mathfrak{M}_p \left\{ \left| \int_0^1 K(x,y)\,\psi_k(y)\,dy \right| \right\},$$

where $q = p/(p-1)$ and $\mathfrak{M}_p$ denotes the mean of power p with respect to the variables x and k. Since $\mathfrak{M}_p$ is an increasing function of p, we have $\mathfrak{M}_p \leqq \mathfrak{M}_2$ for $1 \leqq p \leqq 2$. Thus

$$\sum_{k=1}^{m} \|s_n(\psi_k)\|_p \leqq m \left\{ \frac{1}{m} \sum_{k=1}^{m} \int_0^1 \left| \int_0^1 K(x,y)\psi_k(y)\,dy \right|^2 dx \right\}^{\frac{1}{2}}.$$

Using BESSEL's inequality, we get

$$\sum_{k=1}^{m} \|s_n(\psi_k)\|_p \leqq m \left\{ \frac{1}{m} \int_0^1 \left[\int_0^1 |K(x,y)|^2 dy \right] dx \right\}^{\frac{1}{2}} = m^{\frac{1}{2}} \left\{ \sum_{i=1}^{n} |\lambda_i|^2 \right\}^{\frac{1}{2}} \leqq (mn)^{\frac{1}{2}}.$$

Finally, using MINKOWSKI's inequality, it results that

$$\sum_{k=1}^{m} \|\psi_k\|_p \leqq \sum_{k=1}^{m} \|\psi_k - s_n(\psi_k)\|_p + \sum_{k=1}^{m} \|s_n(\psi_k)\|_p \leqq \alpha_p \sum_{k=1}^{m} M_k + (mn)^{\frac{1}{2}}.$$

This implies (2) and thus proves the lemma.

3. Denote by $C_{r\alpha}$ $(r = 0, 1, \ldots;\ 0 < \alpha \leqq 1)$ the class of all r times continuously differentiable functions $f(x)$ on $[0, 1]$ for which $f^{(r)}(x)$ satisfies a Lipschitz condition of order α (we put $f^{(0)}(x) = f(x)$). For $f \in C_{r\alpha}$ let $N_{r\alpha}(f)$ denote the least Lipschitz constant of order α, i. e.

$$N_{r\alpha}(f) = \underset{x \neq x'}{\text{l. u. b.}} \frac{|f^{(r)}(x) - f^{(r)}(x')|}{|x - x'|^{\alpha}}.$$

We remark that the class C_{r1} consists of those functions which are the $(r+1)$th integral of a bounded measurable function, and

$$N_{r1}(f) = \text{vrai max}\, |f^{(r+1)}(x)|.$$

Theorem I. *There exist positive constants $\gamma_{r\alpha}$ depending on their indices only, such that, for an arbitrarily given orthonormal system of functions on* $[0, 1]$

$$\Phi = \{\varphi_i(x)\} \qquad (i = 1, 2, \ldots),$$

and for an arbitrarily given system of real or complex numbers

$$A = \{\lambda_{ni}\} \qquad (n = 1, 2, \ldots;\ i = 1, \ldots, n)$$

satisfying the condition

$$\sum_{i=1}^{n} |\lambda_{ni}|^2 \leqq n, \tag{3}$$

the quantities

$$\varrho_n(r, \alpha, p) = \underset{f \in C_{r\alpha}}{\text{l. u. b.}} \frac{\|f - s_n(f)\|_p}{N_{r\alpha}(f)},^{2)}$$

where $1 \leqq p \leqq \infty$ *and*

$$s_n(f) = s_n(f; x) = \sum_{i=1}^{n} \lambda_{ni} c_i \varphi_i(x) \qquad (c_i = (f, \varphi_i)), \tag{4}$$

satisfy the inequalities

$$\varrho_n(r, \alpha, p) \geqq \frac{\gamma_{r\alpha}}{n^{r+\alpha}}. \tag{5}$$

Proof. Observe first that, since the mean power (1) is a non-decreasing function of p, we have

$$\varrho_n(r, \alpha, 1) \leqq \varrho_n(r, \alpha, p) \leqq \varrho_n(r, \alpha, \infty) \quad \text{for} \quad 1 \leqq p \leqq \infty. \tag{6}$$

Now consider the functions

$$\psi_k(x) = \sqrt{2} \cos k\pi x \qquad (k = 1, 2, \ldots) \tag{7}$$

forming an orthonormal system on $[0, 1]$. They belong to each of the classes $C_{r\alpha}$ and we have

$$N_{r\alpha}(\psi_k) \leqq \sqrt{2}\, 2^{1-\alpha} (k\pi)^{r+\alpha}.$$

Moreover,

$$\|\psi_k\|_1 = \int_0^1 |\psi_k(x)|\, dx = 2\sqrt{2}/\pi.$$

Applying the Lemma to the systems $\varphi_1, \ldots, \varphi_n$ and $\psi_1, \ldots, \psi_m$ with $m = 2n$, $p = 1$, and $M_k = N_{r\alpha}(\psi_k)$, we get

$$\varrho_n(r, \alpha, 1) \geqq \max_{1 \leqq k \leqq n} \frac{\|\psi_k - s_n(\psi_k)\|_1}{N_{r\alpha}(\psi_k)} \geqq \frac{4\sqrt{2}\, n/\pi - \sqrt{2} n}{\sqrt{2}\, 2^{1-\alpha} \sum_{k=1}^{2n} (k\pi)^{r+\alpha}}.$$

Since

$$\sum_{k=1}^{2n} k^{r+\alpha} < \int_1^{2n+1} u^{r+\alpha}\, du < \frac{(3n)^{r+\alpha+1}}{r+\alpha+1},$$

it results that

$$\varrho_n(r, \alpha, 1) \geqq \frac{\gamma_{r\alpha}}{n^{r+\alpha}}$$

with

$$\gamma_{r\alpha} = \frac{(4/\pi - 1)(r+\alpha+1)}{2^{1-\alpha} \pi^{r+\alpha} 3^{r+\alpha+1}}.$$

Owing to (6), this proves the theorem.

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[2]) Only those f are admitted, for which $N_{r\alpha}(f) > 0$.

4. It is known that there exist constants λ_{ni}, $0 \leqq \lambda_{ni} \leqq 1$, such that if

$$g(x) \sim a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

is the Fourier series of any function $g(x)$ $(-\infty < x < \infty)$ with period 2π) and with $g^{(r)}(x) \in \mathrm{Lip}\,\alpha$, then

$$\left| g(x) - a_0 - \sum_{k=1}^{n-1} \lambda_{nk}(a_k \cos kx + b_k \sin kx) \right| \leqq \frac{A_{r\alpha}}{n^{r+\alpha}} N_{r\alpha}(g)$$

where $N_{r\alpha}(g)$ is the least Lipschitz constant for $g^{(r)}x$ and $A_{r\alpha}$ is a constant independent of g and n.[3]) The same is true of course for the Fourier series of functions of any other period, and in particular for the even functions with period 2, or, which amounts to the same, for the cosine series of the functions $f(x)$ defined on $[0, 1]$, i. e. for the expansion with respect to the orthonormal cosine system

$$\varphi_0(x) = 1, \quad \varphi_k(x) = \sqrt{2} \cos k\pi x \quad (k = 1, 2, \ldots).$$

This means that, for the *cosine system*, and for a *particular choice of Λ*, we have

$$\varrho_n(r, \alpha, p) \leqq \varrho_n(r, \alpha, \infty) = O\left(\frac{1}{n^{r+\alpha}}\right);$$

thus the estimate (5) is the best possible.

The question if the estimate (5) is the best possible even if $\lambda_{ni} = 1$, i. e. in the case when $s_n(f)$ is the nth *partial sum* of the orthogonal expansion of f, remains open except in the case $r = 0$

In the case of the class $C_{0\alpha}$ $(0 < \alpha \leqq 1)$, i e. for the class of the functions $f(x)$ satisfying themselves a Lipschitz condition of order α, the quantities ϱ_n are of the minimal order $O\left(\frac{1}{n^\alpha}\right)$ e. g. for the *orthonormal system of Haar:*

$$\chi_0(x) = 1, \quad \chi_k(x) = \begin{cases} 2^{i/2} & \text{for } \dfrac{2j}{2^{i+1}} < x < \dfrac{2j+1}{2^{i+1}}, \\ 2^{-i/2} & \text{for } \dfrac{2j+1}{2^{i+1}} < x < \dfrac{2(j+1)}{2^{i+1}}, \\ 0 & \text{elsewhere} \end{cases}$$

$$(k = 2^i + j; \quad i = 0, 1, \ldots; \quad j = 0, 1, \ldots, 2^i - 1).$$

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[3]) See, also for further references, B. Sz.-Nagy, Sur une classe générale de procédés de sommation pour les séries de Fourier, *Hungarica Acta Math.*, **1**, n° 3 (1948), 14—52, Theorems IV—VI. We can put e. g. $\lambda_{ni} = \lambda\left(\frac{i}{n}\right)$ with

$$\lambda(u) = \begin{cases} 1 \text{ for } 0 \leqq u \leqq 1/2, \\ 2 - 2u \text{ for } 1/2 \leqq u \leqq 1; \end{cases}$$

in the case $r = 0, \alpha < 1$, we can put also $\lambda_{ni} = 1 - \frac{i}{n}$, i. e. we can consider the Fejér sums.

If

$$s_n(f;x)=\sum_{k=0}^{n-1}(f,\chi_k)\chi_k(x),$$

then[4])

$$f(x)-s_n(f;x)=\frac{1}{|l|}\int_l [f(x)-f(t)\,dt, \tag{8}$$

where $l=l(n,x)$ is a segment containing the point x (eventually as an end-point), of length

$$|l|=2^{-i-1} \quad \text{or} \quad 2^{-i} \quad \text{or} \quad 2^{-i+1},$$

i being determined by the inequalities

$$2^i \leqq n-1 < 2^{i+1}.$$

So we have for $t\in l$ at any case

$$|f(x)-f(t)| \leqq N_{0\alpha}(f)|x-t|^\alpha \leqq N_{0\alpha}(f)2^{(1-i)\alpha} \leqq N_{0\alpha}(f)\,4^\alpha n^{-\alpha},$$

thus, by (8),

$$|f(x)-s_n(f;x)| \leqq N_{0\alpha}(f)\,4^\alpha n^{-\alpha},$$

i. e.

$$\varrho_n(0,\alpha,\infty) \leqq 4^\alpha n^{-\alpha},$$

which, owing to (6), proves our assertion.

5. Let us now consider the class C_r $(r=0,1,\ldots)$ of *all* r times continuously differentiable functions $f(x)$ on $[0,1]$; let $\omega(f^{(r)},\delta)$ be the modulus of continuity of $f^{(r)}(x)$:

$$\omega(f^{(r)},\delta)=\underset{|x-x'|\leqq\delta}{\text{l. u. b.}}\,|f^{(r)}(x)-f^{(r)}(x')|.$$

Theorem II. *There exist positive constants δ_r depending on r only $(r=0,1,\ldots)$, such that, for any orthonormal system of functions $\Phi=\{\varphi_i\}$ and for any system of factors $\Lambda=\{\lambda_{ni}\}$ satisfying* (3), *we have*

$$\varrho_n(r,p)=\underset{f\in C_r}{\text{l. u. b.}}\,\frac{\|f-s_n(f)\|_p}{\omega\left(f^{(r)},\frac{1}{n}\right)} \geqq \frac{\delta_r}{n^r},^{5)} \tag{9}$$

where $s_n(f)$ is defined by (4), *and* $1\leqq p\leqq\infty$.

Proof. Let $\{\psi_k(x)\}$ denote again the system (7); then

$$\omega\left(\psi_k^{(r)},\frac{1}{n}\right)\leqq\sqrt{2}\,(k\pi)^r\frac{k\pi}{n}=\sqrt{2}\,(k\pi)^{r+1}/n.$$

Apply the Lemma to $\varphi_1,\ldots,\varphi_n$; $\psi_1,\ldots,\psi_{2n}$ with $p=1$ and $M_k=\omega\left(\psi_k^{(r)},\frac{1}{n}\right)$.

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[4]) Cf. A. Haar, Zur Theorie der orthogonalen Funktionensysteme, *Inauguraldissertation Göttingen* (1909), pp. 40—43.

[5]) Only those f are concerned, for which $\omega(f^{(r)},\delta)>0$.

It results that

$$\varrho_n(r,p) \geqq \varrho_n(r,1) \geqq \max_{1 \leqq k \leqq n} \frac{\|\psi_k - s_n(\psi_k)\|_1}{\omega\left(\psi_k^{(r)}, \frac{1}{n}\right)} \geqq \frac{4\sqrt{2}\,n/\pi - \sqrt{2}\,n}{\sqrt{2}\sum_{k=1}^{2n}(k\pi)^{r+1}/n} \geqq \frac{\delta_r}{n^r}$$

with

$$\delta_r \geqq \frac{(4/\pi - 1)(r+2)}{\pi^{r+1}\,3^{r+2}}.$$

This concludes the proof.

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