On relatively regular operators.

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1. In this paper I study various classes of bounded linear operators which map a complex Banach space into the whole or part of itself.

The development of this theory has tended to parallel, subject to a retardation of several decades, the development of integral equation theory. Thus the original discoveries of FREDHOLM were extended to abstract spaces by F. RIESZ and many later writers (see [1] for some references), except as regards the theory of the resolvent; extensions of the latter theory have come a good deal later (see for example RUSTON [2]).

In 1921 a new element was introduced into integral equation theory by F. NOETHER [3], who showed that for integral equations of the second kind with kernels of the principal-value type the homogeneous and the transposed homogeneous equations need not necessarily have the same number of linearly independent eigen-functions; the numbers in question were however still both finite, and furthermore the Fredholm solubility conditions remained in force. While the theory of such integral equations was much developed in the following years, the corresponding abstract theory has been discussed only recently. Z. I. HALILOV [4] has given a fairly direct extension of the Noether theory to normed rings. Extensions of the Noether theory along the lines of the theory of linear operators have been given independently by myself [5] and by GOHBERG [6].

As a further example of this parallel development I may cite MIHLIN's criterion (see for example [7]) for the applicability of the Noether theory to a singular integral equation with a parameter, and its interpretation in terms of the normed ring theory of GELFAND (see [5], [8]).

This paper is devoted to the further extension in which of the homogeneous and transposed homogeneous equations only one need necessarily have a finite number of linearly independent solutions. Here it would seem that the abstract theory parts company with integral equation theory, since I do not know of any organised theory of integral equations with such a property. However it is a simple matter to construct examples of linear operators which behave in this way.

2. Let $\mathfrak{R}$ denote a complex Banach space, not necessarily separable or reflexive [1]), and let $\mathfrak{R}^*$ denote the adjoint space of bounded linear functionals, $\mathfrak{R}_1$ the ring of bounded linear operators whose domain is the whole of $\mathfrak{R}$ and whose range is the whole or part of $\mathfrak{R}$. As in my previous papers [1], [5], for any $T \in \mathfrak{R}_1$ I define two functions $\alpha(T)$, $\beta(T)$, the first being the number of linearly independent solutions of $Tf = 0$, $f \in \mathfrak{R}$, the second being the corresponding number for the adjoint equation $lT = 0$, $l \in \mathfrak{R}^*$.

In studying the properties with which I am here concerned, two types of restriction are commonly imposed on operators $T \in \mathfrak{R}_1$, firstly those relating to $\alpha(T)$, $\beta(T)$, and secondly a supplementary restriction on the mapping given by T, which may take various forms, and without which the condition of the first type does not seem to lead to much in the way of results.

As to conditions of the first type, we may list the following three, in order of increasing generality:

First, that found in the work of FREDHOLM, RIESZ, SCHAUDER, HILDEBRANDT and NIKOL'SKIJ, namely

(A_1) $\qquad \alpha(T) < \infty, \quad \beta(T) < \infty, \quad \alpha(T) = \beta(T).$

Next the condition which is investigated in papers by NOETHER, HALILOV, myself and GOHBERG, namely

(A_2) $\quad \alpha(T) < \infty, \quad \beta(T) < \infty, \quad \alpha(T)$ and $\beta(T)$ not necessarily equal.

Finally the situation with which I am mainly concerned in this paper, namely that

(A_3) of the inequalities $\alpha(T) < \infty$, $\beta(T) < \infty$, at least one should hold.

As to the supplementary condition I note first two alternative forms of the condition encountered in the theory of FREDHOLM, RIESZ, SCHAUDER, HILDEBRANDT and NIKOL'SKIJ, namely

(B_1) the inhomogeneous equation $Tf = g$ $(f, g \in \mathfrak{R})$ is soluble for f provided that $lg = 0$ for all $l \in \mathfrak{R}^*$ such that $lT = 0$,

(B_2) the range $T\mathfrak{R}$ of T is closed.

Of these, (B_1) may be regarded as the abstract formulation of FREDHOLM's solubility conditions. Concerning the equivalence of (B_1) and (B_2) see HAUSDORFF [11].

A slightly different condition which I find it convenient to use here is

(B_3) there exists an $X \in \mathfrak{R}_1$ such that $TXT = T$.

Under the latter condition T may be said to be "relatively regular". The

[1]) I observe here that HAMBURGER [9], in his researches on quasi-nilpotent operators on Hilbert spaces, remarks in a footnote on the possibility of the extension of his results to *reflexive* Banach spaces. Points of contact of this paper with that of HAMBURGER will be noted later.

concept of relative regularity is well known in ring theory and has been investigated by I. KAPLANSKY [10].

3. While, as stated, (B_1) and (B_2) are equivalent, the relationship of (B_3) to these two is not entirely clear.

It is readily seen that the case (B_3) is included in (B_1) or (B_2) [2]). Let us assume (B_3) and deduce for example (B_2). Let g_n $(n=1,\ldots)$ be a convergent sequence of elements of $T\mathfrak{R}$, with limit g_0, say. It has then to be shown that $g_0 \in T\mathfrak{R}$. In fact, since $g_n \in T\mathfrak{R}$, we have $TXg_n = g_n$, so that $g_0 = \lim g_n = \lim TXg_n = TX \lim g_n = TXg_0$, which proves the result. It is also a simple matter to deduce (B_1) from (B_3).

The converse, that (B_1) or (B_2) implies (B_3), is at any rate true in certain cases. One such case is that in which (A_2) holds; another is that in which $\mathfrak{R}$ is a Hilbert space, not necessarily separable [3]).

4. I give here certain remarks on relatively regular operators, and list some particular cases.

I observe first that the „relative inverse" X may be chosen so that $TXT=T$, $XTX=X$, so that X also is relatively regular. In fact, if T is relatively regular and X has been found so that $TXT=T$, then writing $X'=XTX$, we have identically $TX'T=T$, $X'TX'=X'$.

I next observe that the class of relatively regular operators includes all operators which are "regular", i. e. which have inverses in $\mathfrak{R}_1$. More generally, this class includes operators with right or left inverses in $\mathfrak{R}_1$.

Less trivially, all finite-dimensional operators are relatively regular. If $T \in \mathfrak{R}_1$ is finite-dimensional, let φ_r $(r=1,\ldots,n)$ be a basis of $T\mathfrak{R}$, with $\varphi_r = Tf_r$ $(r=1,\ldots,n)$. Let there be constructed n linear functionals $l_r \in \mathfrak{R}^*$ $(r=1,\ldots,n)$, such that $l_r(\varphi_s)=1$ $(r=s)$, $=0$ $(r \neq s)$. We then define U for all $f \in \mathfrak{R}$ by $Uf = \sum_1^n l_r(f) f_r$. The operator thus defined belongs to $\mathfrak{R}_1$, and it is easily verified that $TUT=T$, which proves the result.

Less trivially still, if T has the form $S+V$, where $S \in \mathfrak{R}_1$ has an inverse in $\mathfrak{R}_1$ (in particular if $S=I$, the identity operator) and $V \in \mathfrak{R}_1$ is completely continuous (in particular if V is finite-dimensional), then T is relatively regular. This is in substance a result of the RIESZ theory. By this theory, T satisfies the conditions (A_1) and (B_1), (B_2), which then imply (B_3).

Another example of a relatively regular operator is a projection operator, in particular the zero operator.

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[2]) For such a result in Hilbert space see HAMBURGER [9], p. 499.

[3]) Details of the construction are given im my paper [5], or again in [1]; the notation there had T_0, U_0 in place of T, X. See also HAMBURGER [9], p. 498.

I conclude this section with two examples of operators which are not relatively regular. In the first place, a completely continuous operator is not relatively regular unless it is finite-dimensional. For let $T \in \mathfrak{R}_1$ be both completely continuous and relatively regular, with $X \in \mathfrak{R}_1$ as a relative inverse. Let Tf_n $(n = 1, 2, \ldots)$ be any bounded infinite sequence of elements of $T\mathfrak{R}$. This sequence is the same as the sequence $TXTf_n$ $(n = 1, 2, \ldots)$, and this sequence must contain a convergent subsequence, since T is completely continuous and the sequence XTf_n $(n = 1, 2, \ldots)$ is bounded. Thus the linear manifold $T\mathfrak{R}$ has the property that any bounded sequence of its elements contains a convergent subsequence, and by F. RIESZ' converse of the Bolzano—Weierstrass limitpoint theorem, this implies that $T\mathfrak{R}$ is finite-dimensional, as was to be proved.

As a second example, let T belong to an ideal of generalised nilpotent operators; such an ideal is necessarily two-sided. An example of such an operator is given by the integral operator occurring in VOLTERRA's integral equation. Then T cannot be relatively regular. For if X is a relative inverse of T we have, for any positive integral n, $(TX)^n T = T$, showing that $\|(TX)^n\|^{1/n} \geqq 1$, so that TX cannot be generalised nilpotent.

5. In setting up classifications of operators according to these ideas I denote by $\mathfrak{C}(m, n, B_r)$ the set of all $T \in \mathfrak{R}_1$ such that $\alpha(T) = m$, $\beta(T) = n$, and such that T satisfies the supplementary condition (B_r), where r stands for 1, 2 or 3. The cases $m = \infty$, $n < \infty$ and $m < \infty$, $n = \infty$, are not excluded. I shall be concerned mainly with the restriction (B_3); as has been explained, if m and n are both finite, or again if $\mathfrak{R}$ is a Hilbert space, the choice of (B_1), (B_2) or (B_3) is a matter of indifference.

It will further be convenient to denote by $\mathfrak{D}(m, n, B_r)$ the set of all $T \in \mathfrak{R}_1$ such that $\alpha(T) \leqq m$, $\beta(T) \leqq n$, and which satisfy (B_r).

Problems which then arise for discussion may be classified as

(i) *algebraical problems,* such as the nature of products of operators of these classes, or the nature of their factors,

(ii) *topological problems,* such as the result of applying various perturbations to operators of these classes, and whether these classes are open or closed in $\mathfrak{R}_1$,

(iii) *spectral problems,* primarily the study of the regions of the complex λ-plane for which $T - \lambda I$ belongs to one or other of these classes. More generally, we may replace $T - \lambda I$ by an integral function of λ with values in $\mathfrak{R}_1$, or introduce two complex parameters, and so on.

The answers to such questions are largely known in the cases in which both $\alpha(T)$ and $\beta(T)$ are finite; my main purpose here is therefore to carry out certain extensions to all those cases in which at least one of these numbers is finite.

6. In this and the next section I note two simple algebraical properties. The second of these will be needed in what follows, while the first has some bearing on later results and perhaps some intrinsic interest.

Theorem 1. *If* $U, V \in \mathfrak{N}_1$ *and* $I - UV \in \mathfrak{C}(m, n, B_r)$, *then* $I - VU \in \mathfrak{C}(m, n, B_r)$, *where* $r = 1, 2,$ *or* 3.

I first prove the statement as it affects the properties (B_r). (B_1) and (B_2) being equivalent, we consider first (B_2), and have to prove that if $I - UV$ has a closed range, then so has $I - VU$. Let $(I - VU)f_n$ be a convergent sequence of elements of $(I - VU)\mathfrak{N}$, with limit g, say. We have then $U(I - VU)f_n \to Ug$, or $(I - UV)Uf_n \to Ug$, and if $(I - UV)\mathfrak{N}$ is closed we may therefore write $Ug = (I - UV)h$. Hence we derive $g = (I - VU)(g + Vh)$, so that $g \in (I - VU)\mathfrak{N}$, as was to be proved.

As regards (B_3), we have to prove that if $I - UV$ is relatively regular, then so is $I - VU$. Assuming then that there exists an $X \in \mathfrak{N}_1$ such that $(I - UV)X(I - UV) = I - UV$, the desired result follows from the identity $(I - VU)(VXU + I)(I - VU) = I - VU$.

Finally we have to prove the results

$$\alpha(I - UV) = \alpha(I - VU), \quad \beta(I - UV) = \beta(I - VU);$$

the cases in which one or both of these numbers is infinite are not excluded. It will be sufficient to prove the first of these results, that of the other being similar.

Let then $\mathfrak{S}_1, \mathfrak{S}_2$ denote respectively the sets of elements $\varphi, \varphi' \in \mathfrak{N}$ such that $(I - UV)\varphi = 0$, $(I - VU)\varphi' = 0$. We have then $\mathfrak{S}_1 = UV\mathfrak{S}_1$, $\mathfrak{S}_2 = VU\mathfrak{S}_2$, and hence, considering the dimension-numbers, possibly infinite, of these manifolds, $\dim(V\mathfrak{S}_1) = \dim(\mathfrak{S}_1)$, $\dim(U\mathfrak{S}_2) = \dim(\mathfrak{S}_2)$. On the other hand, $(I - UV)\varphi = 0$ implies $(I - VU)V\varphi = 0$, so that $V\mathfrak{S}_1 \subset \mathfrak{S}_2$, and similarly $U\mathfrak{S}_2 \subset \mathfrak{S}_1$; hence we deduce $\dim(V\mathfrak{S}_1) \leqq \dim(\mathfrak{S}_2)$, $\dim(U\mathfrak{S}_2) \leqq \dim(\mathfrak{S}_1)$. From these two pairs of results we have $\dim(\mathfrak{S}_1) \leqq \dim(\mathfrak{S}_2)$, $\dim(\mathfrak{S}_2) \leqq \dim(\mathfrak{S}_1)$, showing that $\dim(\mathfrak{S}_1) = \dim(\mathfrak{S}_2)$, or in other words $\alpha(I - UV) = \alpha(I - VU)$, the result stated.

7. The second result of this kind is

Theorem 2. *If* $T, X \in \mathfrak{N}_1$, *and* $TXT - T \in \mathfrak{C}(m, n, B_r)$, *then* $T \in \mathfrak{D}(m, n, B_r)$, *where* $r = 1, 2$ *or* 3.

Since $T\varphi = 0$, $\varphi \in \mathfrak{N}$ implies $(TXT - T)\varphi = 0$, it is trivial that $\alpha(T) \leqq \alpha(TXT - T)$, and similarly that $\beta(T) \leqq \beta(TXT - T)$, so that all that has to be proved is that if $TXT - T$ satisfies one of the properties (B_1), (B_2) or (B_3), then so does T.

As regards (B_1) or (B_2), it will be sufficient to prove that if $TXT - T$ has a closed range, then so has T. Let then Tf_n be a convergent sequence, with limit g, say. Then $(TXT-T)f_n \to (TX-I)g$, and if $TXT-T$ has a closed range we may therefore write $(TX-I)g=(TXT-T)h$, so that $g=TXg-(TXT-T)h$, showing that $g \in T\mathfrak{R}$, as required.

Finally I prove that if $TXT-T$ is relatively regular, then so is T. Let there exist a Y such that $(TXT-T)Y(TXT-T)=TXT-T$; it then follows that $T\{X-(XT-I)Y(TX-I)\}T=T$, which establishes the result and completes the proof of the theorem.

The particular case of this result which will be used is that if $TXT-T$ is finite-dimensional, then T is relatively regular. This follows from Theorem 2 since, as proved in § 4, finite-dimensional operators are relatively regular.

8. Continuing the subject of algebraic properties of relatively regular operators, I consider in this section conditions under which factors of such operators have the same property, extending some previous results on this subject.

In the result to be proved, and in most subsequent theorems, there is a dual or adjoint form of the result which will at most be enunciated, the proof introducing no new feature.

Theorem 3. *Let $S, T \in \mathfrak{R}_1$ be such that $ST \in \mathfrak{C}(m, n, B_3)$, where $m < \infty$. Then $T \in \mathfrak{D}(m, \infty, B_3)$.*

It is trivial that $\alpha(T) \leqq \alpha(ST) = m$, and thus all that has to be proved is that T is relatively regular.

By hypothesis there exists an $X \in \mathfrak{R}_1$ such that $STXST-ST=0$, i. e. such that $ST(XST-I)=0$. Since by hypothesis $\alpha(ST) < \infty$, it follows that $XST-I$ is finite-dimensional, and hence also $TXST-T$. As pointed out in § 4, such an operator is relatively regular, and by Theorem 2 it follows that T itself is relatively regular, as was to be proved. This proves the theorem.

I may mention that it would also follow that T was relatively regular if instead of $\alpha(ST) < \infty$ we assume $\alpha(S) < \infty$.

The dual result is

Theorem 3'. *Let $S, T \in \mathfrak{R}_1$ be such that $ST \in \mathfrak{C}(m, n, B_3)$, where $n < \infty$. Then $S \in \mathfrak{D}(\infty, n, B_3)$.*

The proof is omitted, being similar to that of Theorem 3.

In particular, if ST is a "generalised Fredholm operator", i. e. satisfies (A_2) and any of the (then equivalent) (B_ν), then both S and T are relatively regular. The special case that if both ST and TS are generalised Fredholm operators, then so are S and T was shown by me in a previous paper [5].

9. I now deduce a result on the products of such operators.

Theorem 4. *Let $S \in \mathfrak{C}(m, n, B_3)$, $T \in \mathfrak{C}(m', n', B_3)$, where $m < \infty$, $m' < \infty$. Then $ST \in \mathfrak{D}(m+m', n+n', B_3)$.*

By hypothesis there exist $U, V \in \mathfrak{N}_1$ such that $SUS - S = 0$, $TVT - T = 0$ Then $S(US - I) = 0$, $T(VT - T) = 0$, and since $\alpha(S) < \infty$, $\alpha(T) < \infty$, we have that $US - I$, $VT - I$ are finite-dimensional.

Writing now

$$VUST = I + (VT - I) + V(US - I)T,$$

it appears that $VUST$ can be put in the form $I + K$, where K is finite-dimensional. This, as mentioned in § 4, is an operator of Fredholm—Riesz type, so that we have $VUST \in \mathfrak{C}(m'', m'', B_3)$ for some $m'' < \infty$. It now follows by Theorem 3 that ST is relatively regular.

It remains to prove that $\alpha(ST) \leqq \alpha(S) + \alpha(T)$, $\beta(ST) \leqq \beta(S) + \beta(T)$; it will be sufficient to give the proof of the first of these. In fact if $ST\varphi = 0$, $\varphi \in \mathfrak{N}$, we must have $T\varphi = \psi$ (say), where $S\psi = 0$. The result now follows from the fact that of these last two equations, the set of solutions of the first is of dimensionality $\alpha(T)$ if the equation is soluble at all, while the set of solutions of the second is of dimensionality $\alpha(S)$. This completes the proof of Theorem 4.

The dual result is

Theorem 4'. *Let* $S \in \mathfrak{C}(m, n, B_3)$, $T \in \mathfrak{C}(m', n', B_3)$, *where* $n < \infty, n' < \infty$. *Then* $ST \in \mathfrak{D}(m + m', n + n, B_3)$.

10. The above result suggest the following problems:

(i) whether the product of two relatively regular operators is itself relatively regular,

(ii) whether the left or right divisors of relatively regular operators are themselves relatively regular.

These results were proved above subject to one-sided restrictions on the dimensionality of the null-manifolds associated with these operators.

Milder conjectures would be that (B_1) or (B_2) could replace (B_3) in Theorems 3 and 4. In essence these theorems would then run as follows (if true):

(iii) if ST has a closed range and $\alpha(ST) < \infty$, then T has a closed range,

(iv) if S and T have closed ranges, and $\alpha(S) < \infty$, $\alpha(T) < \infty$, then ST has a closed range.

11. In an earlier paper [5] I showed that the index $\gamma(T)$, defined by $\gamma(T) = \alpha(T) - \beta(T)$, has under certain conditions the logarithmic property $\gamma(ST) = \gamma(S) + \gamma(T)$. My aim in this section is to extend this property to cases in which the index is, in one or both cases, infinite.

I take first the case in which of $\gamma(S)$, $\gamma(T)$ one is finite, the other being equal to $(-\infty)$. We have

Theorem 5. *Let* $S \in \mathfrak{C}(m, n, B_3)$, $T \in \mathfrak{C}(m', \infty, B_3)$, *where* $m, n, m' < \infty$. *The logarithmic law for the index then holds in the sense that* $ST \in \mathfrak{C}(m'', \infty, B_3)$, $TS \in \mathfrak{C}(m''', \infty, B_3)$, *where* $m'', m''' \leqq m + m'$.

The only parts of this theorem which are not included in Theorem 4 are the statements that $\beta(ST) = \infty$, $\beta(TS) = \infty$. The latter statement follows from the trivial result that $\beta(T) \leqq \beta(TS)$, so that we have only to establish the former.

Now let $l_r \in \mathfrak{R}^*(r = 1, 2, \ldots)$ be an infinite sequence of functionals, any finite set of which are linearly independent, such that $l_r T = 0$. We wish to form a linear combination of the $l_r (r = 1, \ldots, m+1)$, say

$$l_1^* = \sum_{r=1}^{m+1} \alpha_r l_r,$$

the α_r being complex scalars, which admits the representation $l_1^* = k_1 S$ $(k_1 \in \mathfrak{R}^*)$. The latter equation is soluble for k_1 provided that $l_1^* \varphi = 0$ for all $\varphi \in \mathfrak{R}$ such that $S\varphi = 0$; this is the dual of the normal solubility condition (B_1) for S, and is a consequence of (B_3) for S. Let then $\varphi_r \in \mathfrak{R}$ $(r = 1, \ldots, m)$ be a basis of the set of such φ. The condition for solubility may then be written

$$\sum_{r=1}^{m+1} \alpha_r l_r(\varphi_s) = 0 \qquad (s = 1, \ldots, m).$$

This, being a set of m homogeneous equations in the $m+1$ unknowns α_r, must have a non-trivial solution. Thus k_1 can be found, and is not the zero functional. In a similar way we can derive a second functional k_2 from the $l_r (r = m+2, \ldots, 2m+2)$, and so on indefinitely. Moreover, any finite set of the k_r will be linearly independent, since this is so for the l_r. Since all the k_r are such that $k_r ST = 0$, it follows that $\beta(ST) = \infty$, as asserted.

The dual result is

Theorem 5'. *Let* $S \in \mathfrak{C}(m, n, B_3)$ *and let* $T \in \mathfrak{C}(\infty, n' B_3)$, *where* $m, n, n' < \infty$. *Then* $ST \in \mathfrak{C}(\infty, n'', B_3)$, $TS \in \mathfrak{C}(\infty, n''', B_3)$, *where* $n'', n''' \leqq n + n'$.

This is of course the case in which $\gamma(S)$ is finite and $\gamma(T)$ equals $+\infty$.

The cases in which $\gamma(S)$ and $\gamma(T)$ are both equal to $+\infty$ or to $-\infty$ are trivial. The results are

Theorem 6. *Let* $S \in \mathfrak{C}(m, \infty, B_3)$, $T \in \mathfrak{C}(m', \infty, B_3)$, *where* $m, m' < \infty$. *Then* $ST \in \mathfrak{C}(m'', \infty, B_3)$, *where* $m'' \leqq m + m'$.

Theorem 6'. *Let* $S \in \mathfrak{C}(\infty, n, B_3)$, $T \in \mathfrak{C}(\infty, n', B_3)$, *where* $n, n' < \infty$. *Then* $ST \in \mathfrak{C}(\infty, n''. B_3)$, *where* $n'' \leqq n + n'$.

For Theorem 6, for instance, statement that $\beta(ST) = \infty$ follows from the elementary result $\beta(ST) \geqq \beta(S)$, and similarly for Theorem 6'.

Finally, as a consequence, we have a result on the case in which $\gamma(S)$ and $\gamma(T)$ are infinite with opposite signs.

Theorem 7. *Let $ST \in \mathfrak{C}(m, n, B_3)$, where $m, n < \infty$. The logarithmic law then holds in the sense that if $\gamma(S) = +\infty$, then $\gamma(T) = -\infty$, and conversely.*

Assume say that $\gamma(S) = +\infty$, say $S \in \mathfrak{C}(\infty, n', B_2)$, where $n' < \infty$. Since $\alpha(T) \leqq \alpha(ST) < \infty$, $\gamma(T)$ can be either finite or equal to $-\infty$; it cannot equal $+\infty$. We have therefore only to reject the possibility of $\gamma(T)$ being finite, or in fact the possibility of $\beta(T)$ being finite. Suppose then if possible that $T \in \mathfrak{C}(m', n'', B_3)$, where $m', n'' < \infty$. It would then follow by Theorem 5′ that $m = \alpha(ST) = \infty$, contrary to hypothesis. This proves the result. The converse assertion may be proved similarly.

12. I now pass to problems of the second type, namely those concerning the neighbourhoods of an operator of the type considered. There are three types of perturbation to be considered, firstly perturbation by the addition of a general element of $\mathfrak{R}_1$ of suitably small norm, secondly perturbation by the addition of a completely continuous operator, not necessarily small, and lastly perturbation by the addition of an operator which is small and which depends analytically upon a scalar complex parameter.

In this section I treat the case of a small general perturbation. I note first a previous result of mine [5], that if $T \in \mathfrak{R}_1$ is such that $\alpha(T)$ and $\beta(T)$ are both finite and T satisfies (B_1), or of course (B_2) or (B_3), then for $T' \in \mathfrak{R}_1$ lying in a neighbourhood of T the same conditions hold, the index being thereby stable, so that $\gamma(T') = \gamma(T)$. The additional information that $\alpha(T') \leqq \alpha(T)$, together with an extension to unbounded operators, has been given by Kreĭn and Krasnosel'skij [12]; see also B. Sz.-Nagy [13].

Here I wish to extend such results to the case in which only one of $\alpha(T), \beta(T)$ is assumed to be finite[4]). I prove first

Theorem 8. *Let $T \in \mathfrak{C}(m, n, B_3)$ where at least one of m, n is finite. Then there exists a positive number ϱ, such that if $T' \in \mathfrak{R}_1$, $\|T' - T\| < \varrho$, then $T' \in \mathfrak{D}(m, n, B_3)$.*

By hypothesis there exists an $X \in \mathfrak{R}_1$ such that $TXT - T = 0$. Suppose first that $m = \alpha(T) < \infty$. Since $T(XT - I) = 0$ it follows that $XT - I$ is finite-dimensional, of dimensionality not exceeding m.

Write now $A = T' - T$, where $\|A\| < \varrho$, and ϱ is to be chosen later. Following the argument of § 5 of my previous paper [1] I write

$$XT' = (I + XA) + (XT - I),$$

whence it appears that if we take $\varrho < \|X\|^{-1}$, then XT' can be represented

[4]) In the above-cited paper [11] Kreĭn and Krasnosel'skij refer to a previous paper (of which I have unfortunately not yet able to get a copy) by them and D. P. Mil'man [14] where there appears a partial extension along these lines, to the effect that if $\alpha(T) = 0$, then $\beta(T') = \beta(T)$, even if $\beta(T)$ is unbounded.

as the sum of an operator with an inverse in $\mathfrak{R}$, and a finite-dimensional operator, and, as mentioned in § 4, forms a trivial case of the Riesz theory. Thus by Theorem 3 it follows that T' is relatively regular.

It remains to prove that $\alpha(T') \leqq \alpha(T) = m$, the latter being assumed finite. As in § 5 of [1], this follows from the equation

$$(I+XA)^{-1}XT' = I + (I+XA)^{-1}(XT-I).$$

The argument is of course similar if we assume $n = \beta(T) < \infty$.

Next I prove the result regarding stability of the index.

Theorem 9. *Let $T \in \mathfrak{C}(m, \infty, B_3)$, where $m < \infty$, so that $\gamma(T) = -\infty$. Then there is a positive ϱ such that if $T' \in \mathfrak{R}, \|T'-T\| < \varrho$, then $T' \in \mathfrak{C}(m', \infty, B_3)$, where $m \leqq m$, so that $\gamma(T') = -\infty$.*

In view of the result of Theorem 8, it is only necessary to prove that $\beta(T') = \infty$. As in the proof of Theorem 8, $XT-I$ is finite-dimensional, so that XT is of Fredholm-Riesz type. Since $\gamma(T) = -\infty$, we deduce that $\gamma(X) = \infty$. Furthermore, again as in the proof of Theorem 8, XT' will also be of Fredholm-Riesz type, and applying Theorem 7 once more we deduce from the fact that $\gamma(X) = \infty$ the result that $\gamma(T') = -\infty$, showing that $\beta(T') = \infty$, as required.

13. I now consider perturbations by a completely continuous operator, not necessarily small. In [5] I have proved the result that if $T \in \mathfrak{C}(m, n, B_1)$, where $m, n < \infty$, and $V \in \mathfrak{R}$, is completely continuous, then $T + V \in \mathfrak{C}(m', n', B_1)$, where $m', n' < \infty$ and $\gamma(T+V) = \gamma(T)$. KREIN and KRASNOSEL'SKIJ [11] have given a result in some ways more general than this, in that T need not be bounded, but more special in other ways, in particular in that the perturbing operator is to be finite-dimensional instead of completely continuous; see however B. SZ.-NAGY [13]. I now give an extension to the case of an infinite index, namely

Theorem 10. *Let $T \in \mathfrak{C}(m, n, B_3)$, where at least one of m, n is finite, and let $V \in \mathfrak{R}_1$ be completely continuous. Then $T + V \in \mathfrak{C}(m', n', B_3)$, where $m' < \infty$ if $m < \infty$, and $n' < \infty$ if $n < \infty$.*

As before there exists $X \in \mathfrak{R}_1$ such that $TXT - T = 0$. Assuming first that $m = \alpha(T) < \infty$, we have that $XT - I$ is finite-dimensional. Writing

$$X(T+V) = I + XV + (XT-I),$$

it appears that $X(T+V)$ can be represented as the sum of the identity operator, a completely continuous operator, and a finite-dimensional operator, so that $X(T+V)$ is an operator of Fredholm-Riesz type. Thus by Theorem 3 it follows that $T+V$ is relatively regular, and also that $\alpha(T+V) < \infty$.

The case in which $n = \beta(T) < \infty$ is of course discussed similarly.

As regards the stability of the index we have

Theorem 11. *Let $T \in \mathfrak{C}(m, \infty, B_3)$, where $m < \infty$, so that $\gamma(T) = -\infty$, and let $V \in \mathfrak{R}_1$ be completely continuous. Then $T + V \in \mathfrak{C}(m', \infty, B_3)$, where $m' < \infty$, so that $\gamma(T+V) = -\infty$.*

The proof, like that of Theorem 9, proceeds by application of Theorem 7 to the operators $XT, X(T+V)$.

The result can be dualised in an obvious way.

14. Before proceeding to analytic perturbations, I remark that the results of the last two sections are in some ways best possible results.

We have shown for example that the set of all $T \in \mathfrak{R}_1$ such that at least one of $\alpha(T), \beta(T)$ is finite and such that T is relatively regular is an open set in $\mathfrak{R}_1$. If we relax the restriction on $\alpha(T), \beta(T)$ the result would become that the set of all $T \in \mathfrak{R}_1$ which are relatively regular is open. This however is true only if $\mathfrak{R}$ is finite-dimensional. For the zero operator is relatively regular, and if all T in a neighbourhood of zero of the form $\|T\| < \varrho$ were also to be relatively regular, then so would all $T \in \mathfrak{R}_1$, by multiplication by a suitable scalar. This however is only possible when $\mathfrak{R}$ is finite-dimensional [5]), since otherwise we could construct a $T \in \mathfrak{R}_1$ which was completely continuous without being finite-dimensional, and which would therefore not be relatively regular, as mentioned in § 4.

In the same way the statement that if T is relatively regular and V is completely continuous, then $T+V$ is also relatively regular, is not necessarily true without the restriction that at least one of $\alpha(T), \beta(T)$ should be finite. This again is shown by the particular case $T=0$. We can however assert that if T is relatively regular and K is finite-dimensional, then $T+K$ is relatively regular. This follows easily from Theorem 2, since if $TXT - T = 0$, then $(T+K)X(T+K) - (T+K)$ is finite-dimensional. We may regard this as a consequence of the fact that the finite-dimensional operators form an ideal (left and right) of relatively regular operators in the ring $\mathfrak{R}_1$, though whether there are any other such ideals which are not trivial is not obvious.

15. The rest of this paper is devoted to the case in which $T = T_\lambda$ depends analytically upon a complex scalar parameter λ. In this section I consider the behaviour of T_λ in the small; the basic fact to be established is that in the neighbourhood of any λ-value, under certain restrictions, the functions $\alpha(T_\lambda)$ and $\beta(T_\lambda)$ take constant values, not exceeding their values at the λ-value in question. In the case in which both values are finite this result is known [6]). Here therefore I give the extension to the case in which only one

[5]) See Kaplansky [10].

[6]) See for instance my previous paper [1]: the argument there was applied to a case in which α and β had equal values and in which T_λ had a special polynomial form, but applies more generally. See also the papers of Gohberg cited in [1].

of α and β need be finite. Without loss of generality we may take the perturbation to be about $\lambda = 0$. I prove now

Theorem 12. *Let $T \in \mathfrak{C}(m, \infty, B_3)$, where $m < \infty$, and let A_λ be an analytic function of λ, with values in $\mathfrak{N}_1$, defined in a neighbourhood of $\lambda = 0$ and vanishing at $\lambda = 0$. Then there is a positive number ϱ and a non-negative integer m' with $m' \leqq m$, such that for $0 < |\lambda| < \varrho$ we have $T - A_\lambda \in \mathfrak{C}(m', \infty, B_3)$.*

Taking $X \in \mathfrak{N}_1$ such that $TXT = T$, and writing $P_1 = I - XT$, so that P_1 is of finite dimensionality not exceeding m, we derive the equation

$$(I - XA_\lambda)^{-1} X(T - A_\lambda) = I - (I - XA_\lambda)^{-1} P_1,$$

valid at any rate if $(I - XA_\lambda)$ has an inverse, and so in a λ-region of the form $|\lambda| < \sigma$ for some positive σ. The reasoning by which it is deduced that $\alpha(T - A_\lambda)$ takes a constant value, not exceeding $\alpha(T)$, in a region of the form $0 < |\lambda| < \varrho$, has already been given in §§ 5—7 of [1].

The remaining assertions of this theorem, that for $0 < |\lambda| < \varrho$ we have that $T - A_\lambda$ is relatively regular with $\beta(T - A_\lambda) = \infty$ follow from Theorem 9. This completes the proof of Theorem 12.

Theorem 12 can of course be dualised.

16. I now pass to deductions regarding the behaviour in the large. With the assumptions and notation of Theorem 12, let us consider the maximal connected region which includes $\lambda = 0$ and such that for every λ-value in this region $T - A_\lambda$ exists, is relatively regular and furthermore $\alpha(T - A_\lambda) < \infty$. By Theorem 12, this λ-region is non-empty and is open in the complex λ-plane.

The basic fact now to be established is that in this region we have almost everywhere $\alpha(T - A_\lambda) = m'$, $\beta(T - A_\lambda) = \infty$; more precisely, we have $\beta(T - A_\lambda) = \infty$ everywhere in this region, and $\alpha(T - A_\lambda) = m'$ everywhere except possibly at isolated points, with no limit point in this region, at which $m' < \alpha(T - A_\lambda) < \infty$. Such isolated points form a natural generalisation of the concept of an eigen-value. We have by Theorem 12 that $\alpha(T - A_\lambda) = m'$, $\beta(T - A_\lambda) = \infty$ in a region of the form $0 < |\lambda| < \varrho$; the extension to the whole λ-region with the above-noted exceptional points is achieved by the argument given for a more special case in § 8 of [1].

It may happen that this λ-region covers the entire λ-plane. This will for example be the case if T_λ has the form $T + \sum_1^\infty \lambda^r V_r$, where T satisfies the conditions of Theorem 12, the V_r are completely continuous, and the series is absolutely convergent for all λ; a more special case of this kind was the subject of [1]. Another example with this property would be $T_\lambda = T - \lambda A$, where T satisfies the conditions of Theorem 12 and A belongs to an ideal of generalised nilpotent operators.

17. I now discuss the notions in spectral theory which accord best with the ideas of the present paper. Let as before T_λ denote an analytic function of λ with values in $\mathfrak{N}_1$; for simplicity we will take T_λ to be an integral function. The standard case $T_\lambda = T - \lambda I$ is of course included.

Let us denote by $\mathfrak{S}(B_3)$ the set of all λ-values for which T_λ is relatively regular, and by $\mathfrak{S}_0(B_3)$ the subset of $\mathfrak{S}(B_3)$ of λ-values for which in addition at least one of $\alpha(T_\lambda), \beta(T_\lambda)$ is finite. By Theorem 8, or Theorem 12, $\mathfrak{S}_0(B_3)$ is an open set; the example $T_\lambda = \lambda V$, where V is completely continuous, shows that $\mathfrak{S}(B_3)$ need not be an open set.

We know that every λ-value in $\mathfrak{S}_0(B_3)$ is an interior point of an open λ-region in which the functions α and β take constant values almost everywhere. The case in which one of these values is infinite was examined in the last two sections; as already mentioned, the case in which both values are finite has been dealt with in previous papers. Our procedure is then to divide up $\mathfrak{S}_0(B_3)$ into these open regions.

Let then m, n denote a pair of numbers, admissible values for which are zero, any positive integer, or $+\infty$, with the proviso that at least one of m and n must be finite. Corresponding to any such pair m, n there can exist at most a denumerable sequence of open connected regions of the λ-plane, which we denote by $\mathfrak{S}_r(m, n, B_3)$ $(r = 1, 2, \ldots)$, and with the property that in any one of them $T_\lambda \in \mathfrak{C}(m, n, B_3)$, with the possible exception of at most a denumerable sequence of isolated points, the generalised eigen-values, at which $T_\lambda \in \mathfrak{C}(m+k, n+k, B_3)$, for some varying positive k.

We show later how within any such region $\mathfrak{S}_r(m, n, B_3)$ we may associate with T_λ certain meromorphic functions with singularities at the generalised eigen-values.

18. Before proceeding to a detailed analysis of one of these regions $\mathfrak{S}_r(m, n, B_3)$ it will perhaps be useful to compare the above system of spectral classification with other known classifications.

Closely related to the above classification are the concept of the Fredholm region and its generalisations. The Fredholm region, introduced by NIKOL'SKIJ (see [1] for reference) in the case in which T_λ depends linearly on λ, will consist of the totality of sets $\mathfrak{S}_r(m, m, B_3)$, where $m < \infty$, i. e. the set of λ for which $\alpha(T_\lambda) = \beta(T_\lambda) < \infty$ and for which T_λ is relatively regular or, what comes to the same thing, satisfies (B_1) or (B_2). The corresponding region in which $\alpha(T_\lambda)$ and $\beta(T_\lambda)$ are both finite but not necessarily equal was termed by me ([5], p. 11) the "generalised Fredholm region" and by GOHBERG (see [1] for reference) the "Noether region". Here of course I am concerned with a still more general region, in which only one of $\alpha(T_\lambda), \beta(T_\lambda)$ need be finite, T_λ still being relatively regular.

Rather less appropriate for the present purpose are the notions of spectral classification which arise naturally in the spectral theory of self-adjoint operators on Hilbert space. Taking for example the formulations of HILLE ([15], pp. 31, 97) and translating them into the terminology of the present paper we should classify the λ-values as follows:

(i) resolvent set, such that $T_\lambda \in \mathfrak{C}(0, 0, B_3)$,

(ii) spectrum, such that T_λ does not belong to $\mathfrak{C}(0, 0, B_3)$.

The spectrum would then be subdivided as follows:

(iii) point spectrum, such that $\alpha(T_\lambda) > 0$, whether T_λ is relatively regular or not,

(iv) residual spectrum, such that $\alpha(T_\lambda) = 0, \beta(T_\lambda) > 0$, again whether T_λ is relatively regular or not,

(v) continuous spectrum, such that $\alpha(T_\lambda) = \beta(T_\lambda) = 0$, but such that T_λ is not relatively regular.

A further definition would be that points belonging to the "point spectrum" would be termed "characteristic values".

To illustrate the effect of these classifications by examples, a set of the form $\mathfrak{S}_r(0, 1, B_3)$ would go into the residual spectrum except for what we have called generalised eigen-values which would go into the point spectrum; a set of the form $\mathfrak{S}_r(1, 0, B_3)$, or of the form $\mathfrak{S}_r(1, 2, B_3)$ would go entirely into the point spectrum. The classification would thus obscure the essential similarities between the various regions $\mathfrak{S}_r(m, n, B_3)$, and in addition would lump together operators which are relatively regular with those that are not.

Certain classifications introduced by HAMBURGER [9] should also be mentioned for comparison. HAMBURGER considers the spectral character of an operator with respect to a subspace $\mathfrak{M}$ of the Hilbert space $\mathfrak{H}$; here I translate some of his definitions into the terms of the present paper in the special case in which $\mathfrak{M}$ coincides with $\mathfrak{H}$. An improper eigen-value is a value λ such that $\alpha(T_\lambda) > 0, \beta(T_\lambda) = 0$, whether T_λ is relatively regular or not. The set of such λ's forms the co-residual spectrum, the adjoint concept to the residual spectrum. If, I now observe, we revise the definition (iii) of the point spectrum to be that $\alpha(T_\lambda) > 0$ and $\beta(T_\lambda) > 0$, we obtain a partition of the spectrum which is at any rate symmetrical as between $\alpha(T_\lambda)$ and $\beta(T_\lambda)$, but which does not entirely remove the above objections.

A further definition of HAMBURGER is that λ is to be called a point of the first or the second kind according as T_λ has or has not a closed range. In the present terminology I would write the set of points of the first kind as $\mathfrak{S}(B_2)$. Since relative regularity implies the closed range property we have, for a Banach space, $\mathfrak{S}(B_2) \supseteq \mathfrak{S}(B_3)$, the two sets coinciding if $\mathfrak{R}$ is a Hilbert space. For the case $T_\lambda = A - \lambda I$, HAMBURGER states (loc. cit. p. 505) that the set of points of the first kind form an open set.

19. I now pass to the theory of the analytic functions which may be associated with T_λ in a particular region $\mathfrak{S}_r(m, n, B_3)$; as previously, at least one of m, n must be finite. These functions are three in number, one forming a generalisation of the resolvent, and two being projection operators characterising the left and right null-manifolds of T_λ, in cases of course in which $m, n > 0$.

Assume then that T is an integral function of λ with values in $\mathfrak{R}_1$, and let λ_0 be a point of one of the corresponding λ-regions $\mathfrak{S}_r(m, n, B_r)$, which region we denote for brevity by $\mathfrak{S}_0$. Varying the notation of § 15, I write $T_\lambda = T_{\lambda_0} - A_\lambda = T_0 - A$, so that $A = 0$ when $\lambda = \lambda_0$. I assume that λ_0 is not a generalised eigen-value, so that $\alpha(T_0) = m, \beta(T_0) = n$; this restriction has sense of course only if m and n are not both infinite. Let X_0 be a relative inverse of T_0, so that $T_0 X_0 T_0 = T_0$, $X_0 T_0 X_0 = X_0$. Write also $P_{10} = I - X_0 T_0$, $P_{20} = I - T_0 X_0$, so that P_{10} and P_{20} are projection operators whose ranges on $\mathfrak{R}$ and $\mathfrak{R}^*$ are the right and left null-manifolds of T_0.

The problem is then to find analytic expressions for $X_\lambda, P_{1\lambda}, P_{2\lambda}$ which fulfil the same roles for T_λ, for all $\lambda \in \mathfrak{S}_0$, except possibly at the generalised eigen-values in $\mathfrak{S}_0$. It is easily shown that for λ in a neighbourhood of λ_0 such functions are given by

$$X_\lambda = (I - X_0 A)^{-1} X_0 = X_0 (I - A X_0)^{-1}, \tag{19. 1}$$

$$P_{1\lambda} = I - X_\lambda T_\lambda = (I - X_0 A)^{-1} P_{10}, \tag{19. 2}$$

$$P_{2\lambda} = I - T_\lambda X_\lambda = P_{20} (I - A X_0)^{-1}. \tag{19. 3}$$

Concerning these definitions I remark firstly that they have sense for all λ such that $(I - X_0 A)$ and $(I - A X_0)$ have inverses in $\mathfrak{R}_1$. Secondly, these two conditions are equivalent, that is to say $(I - X_0 A)$ and $(I - A X_0)$ will both or neither have inverses; this is a rather special case of Theorem 1. Thirdly I remark that the equivalence of the alternative forms given in each of (19. 1-3) can be verified by simple calculations, assuming that $(I - A X_0)$, $(I - X_0 A)$ have inverses.

More precisely, we can set up a spectral classification of λ-values into open connected regions $\mathfrak{S}_r'(m, n, B_3)$ for the operators $I - X_0 A$, $I - A X_0$, the regions being identical for these two operator-functions by Theorem 1. My main concern here is with the particular one of these regions which contains λ_0. Since these operators have inverses near λ_0, this region will be of the form $\mathfrak{S}_r'(0, 0, B_3)$, and for brevity I denote it by $\mathfrak{S}_0'$. These two operators will then have inverses throughout $\mathfrak{S}_0'$, with the possible exception of isolated points with no limit-point in $\mathfrak{S}_0'$, which will be eigen-values in the ordinary sense for $I - A X_0$ and $I - X_0 A$.

20. What I wish to prove here is then that if X_λ is defined by (19. 1) then X_λ provides a relative inverse of T_λ for all $\lambda \in \mathfrak{S}_0'$ with the exception of eigen-values of $I - A X_0$, $I - X_0 A$; these two operator-functions will have the same eigen-values. It has therefore to be proved that for such λ-values

(20. 1-2) $$T_\lambda X_\lambda T_\lambda = T_\lambda, \quad X_\lambda T_\lambda X_\lambda = X_\lambda.$$

Before proving this it is necessary to clarify the relationship between $\mathfrak{S}_0$ and $\mathfrak{S}_0'$. Assuming, as I do, that at least one of m, n is finite, then it can be asserted that $\mathfrak{S}_0' \subseteq \mathfrak{S}_0$. For let $\lambda \in \mathfrak{S}_0'$, so that $I - X_0 A$ is relatively regular and $\alpha(I - X_0 A) = \beta(I - X_0 A)$, both numbers being finite. Assume for definiteness that $m < \infty$, so that P_{10} is finite-dimensional. We have the equation $X_0 T = X_0(T_0 - A) = (I - X_0 A) - P_{10}$. An application of Theorem 10 then shows that $X_0 T$ is relatively regular and that $\alpha(X_0 T_\lambda) < \infty$. It now follows by Theorem 3 that T_λ is relatively regular and that $\alpha(T_\lambda) < \infty$. Thus $\mathfrak{S}_0'$ is a connected region throughout which T_λ is relatively regular with $\alpha(T_\lambda) < \infty$, and furthermore $\mathfrak{S}_0'$ contains λ_0. But $\mathfrak{S}_0$ is the maximal such region containing λ_0, so that $\mathfrak{S}_0' \subseteq \mathfrak{S}_0$, as was to be proved. The argument is of course similar if only $n < \infty$.

The position becomes simpler if both m and n are finite, for then $\mathfrak{S}_0'$ and $\mathfrak{S}_0$ coincide. In this case, if $\lambda \in \mathfrak{S}_0$, X_0 and T_λ will both satisfy conditions (A_2) and (B_3) of § 2, so that by Theorem 4 of this paper (or by Theorem 2 of [5]) $X_0 T_\lambda$ will satisfy the same conditions, and hence also $(I - X_0 A)$, by Theorem 10. It now follows that $\mathfrak{S}_0$ is a connected region, containing λ_0, in which $\alpha(I - X_0 A)$ and $\beta(I - X_0 A)$ are finite and $(I - X_0 A)$ is relatively regular, so that $\mathfrak{S}_0 \subseteq \mathfrak{S}_0'$, which together with the previous result shows that the two regions coincide. It does not of course follow that the generalised eigen-values in the two regions coincide.

Reverting to the previous case, let us only assume that $m < \infty$, and let λ be a point of $\mathfrak{S}_0'$ which is not a singularity of $(I - X_0 A)^{-1}, (I - A X_0)^{-1}$. I wish under these assumptions to justify (20. 1-2). Of these, (20. 2) is trivial since

$$X_\lambda - X_\lambda T_\lambda X_\lambda = X_\lambda (I - T_\lambda X_\lambda) = (I - X_0 A)^{-1} X_0 P_{20} (I - A X_0)^{-1},$$

by (19. 1) and (19. 3), and this vanishes since $X_0 P_{20} = X_0 (I - T_0 X_0) = 0$. Considering now (20. 1) we have by (19. 2)

(20. 3) $$T_\lambda (I - X_\lambda T_\lambda) = T_\lambda (I - X_0 A)^{-1} P_{10},$$

and it has to be proved that the right-hand side vanishes. Consider the set of $\varphi \in \mathfrak{R}$ such that $T\varphi = 0$. For such φ, by (19.2) we have $\varphi = (I - X_0 A)^{-1} P_{10} \varphi$, so that the linear manifold of such φ is contained in the range of the (projection) operator $(I - X_0 A)^{-1} P_{10}$. But this operator has the same dimensionality as P_{10}, which is by hypothesis $m < \infty$. Furthermore by the hypotheses the set of such φ is of dimensionality not less than m, the minimum value of $\alpha(T_\lambda)$ throughout the region $\mathfrak{S}_0$. It follows that the set of such φ is of dimensionality precisely m, and that the set coincide with the range of $(I - X_0 A)^{-1} P_{10}$, so that the right-hand side of (20. 3) vanishes.

Summing up we have proved that if $\lambda \in \mathfrak{S}_0'$ is not an eigen-value of $(I - X_0 A)$, $(I - A X_0)$, then it is not a generalised eigen-value of T_λ, and furthermore the relative inverse of T_λ is given by (19. 1). It follows from this that for such λ the projection operators characterising the right and left null-manifolds of T_λ are given by (19. 2-3).

The argument is of course similar if instead of $m < \infty$ we assume that $n < \infty$.

While we have proved that the generalised eigen-values of T_λ are included in the (in this case ordinary) eigen-values of either of $(I - X_0 A)$, $(I - AX_0)$, the possibility remains that the latter two operators might have eigen-values which were not generalised eigen-values of T_λ. If m and n are not both zero the relative inverse is of course not unique, which suggests the question of whether in this case X_0 can be chosen so as to make the two sets of singularities identical. If $m = n = 0$, so that T_0 has an inverse which is also the relative inverse, it is readily seen that the two sets are the same.

21. I conclude this paper by illustrating some of these concepts in a simple case in which only one of the indices α, β is finite [7]).

Consider the space of sequences of complex numbers

$$f = (f_0, f_1, \ldots)$$

with the norm, say, $\|f\| = \sum_0^\infty |f_n|$, and the bounded linear transformation

$$Tf = (f_0, 0, f_1, 0, f_2, 0, \ldots) \tag{21. 1}$$

for which obviously $\alpha(T) = 0$, $\beta(T) = \infty$. A relative inverse, in fact in this case a left inverse, is given by

$$Xf = (f_0, f_2, f_4, \ldots), \tag{21. 2}$$

so that $XT = I$, $TXT = T$, $XTX = X$.

I now consider two examples of spectral theory involving this operator. Take first $T_\lambda = T - \lambda I$, where T is given by (21.1). Since $\|T\| = 1$ it is clear that T_λ has an inverse for $|\lambda| > 1$. We have further, if X is given by (21.2), $XT_\lambda = I - \lambda X$, and since $\|X\| = 1$ it follows that for $|\lambda| < 1$ a left inverse, and so a relative inverse, of T_λ is given by $(I - \lambda X)^{-1} X$. Thus the whole λ-plane is divided up into the following three regions: (i) the region $|\lambda| > 1$, which will be of type $\mathfrak{S}(0, 0, B_3)$, (ii) the region $|\lambda| < 1$, of type $\mathfrak{S}(0, \infty, B_3)$, (iii) the region $|\lambda| = 1$, which does not lie in the region I termed $\mathfrak{S}_0(B_3)$, where T_λ is relatively regular and at least one of $\alpha(T_\lambda), \beta(T_\lambda)$ is finite. In this case there are no eigen-values, either generalised or ordinary, though these may be introduced by making slight modifications to the operator T.

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[7]) A similar example in which α and β are both finite but different has been briefly considered by Beurling [16], p. 242, Hamburger [9], p. 504.

As a second example I take $T_\lambda = T - \lambda A$, where T is given as before by (21.1), and A is given by

$$Af = (f_0, f_0, f_1, f_1, f_2, \ldots).$$

We have then

$$T_\lambda f = (f_0(1-\lambda), -\lambda f_0, (1-\lambda)f_1, -\lambda f_1, \ldots),$$

from which it is clear that a "relative resolvent" X_λ, which in this case is a left inverse and is independent of λ, is given by

$$X_\lambda f = (f_0 - f_1, f_2 - f_3, \ldots).$$

It follows that in this case the whole λ-plane is a region of the form $\mathfrak{S}(0, \infty, B_3)$, and again there are no generalised eigen-values.

Apropos of the remarks at the end of § 20, I now show that for the last example the "relative resolvent" may be chosen so that it has singularities which are not generalised eigen-values of T_λ. We take the left inverse of T given by

$$X'f = (f_0 - a_0 f_1, f_2 - a_1 f_3, \ldots),$$

where the a_r form any bounded sequence of complex numbers. We have then $X'T_\lambda = X'(T - \lambda A) = I - \lambda X'A$, and the corresponding relative resolvent is given by $(I - \lambda X'A)^{-1}X'$, provided that $(I - \lambda X'A)$ has an inverse. However we have

$$X'Af = ((1-a_0)f_0, (1-a_1)f_1, \ldots),$$

so that the singularities of $(I - \lambda X'A)^{-1}$ will be at the points $\lambda = (1-a_r)^{-1}$, $(r = 0, 1, \ldots)$. We may of course choose the a_r so as to make these singularities dense along a closed curve, or dense inside a region, and we shall then have cases in which the connected region in which $(I - \lambda X'A)^{-1}$ exists, denoted in the notation of § 20 by $\mathfrak{S}_0'$, is a proper sub-region of $\mathfrak{S}_0$, the connected region in which T_λ is relatively regular with at least one of $\alpha(T_\lambda), \beta(T_\lambda)$ finite; this, it will be recalled, was shown to be impossible in cases in which $\alpha(T_\lambda)$, $\beta(T_\lambda)$ are both finite.

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