On a lemma of Stieltjes on matrices.

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Notations.

$a, b, c, \ldots$ scalars	$\mathbf{A}^* =$ transposed of $\mathbf{A}$
$\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ column vectors	$\lvert\mathbf{A}\rvert =$ determinant of $\mathbf{A}$
$\mathbf{a}^*, \mathbf{b}^*, \mathbf{c}^*, \ldots$ row vectors	$\langle a_1, a_2, \ldots, a_n \rangle =$ diagonal matrix
$\mathbf{A}, \mathbf{B}, \mathbf{C}, \ldots$ matrices	$\mathbf{E} = \langle 1, 1, \ldots, 1 \rangle =$ unit matrix

In an article published in 1886,[1]) STIELTJES gave as a lemma a theorem amounting to the following:

Theorem I. *If the elements off the principal diagonal of the matrix of a positive definite quadratic form are all negative, then all the elements of the inverse of that matrix are positive.*

An extension of STIELTJES' remark to include not necessarily symmetric matrices has been made by J. L. MOSAK[2]) and H. E. GOHEEN[3]) in a lemma which amounts to the following:

Theorem II. *If*

α) all the principal minors of a matrix are positive, and

β) all the elements off its main diagonal are negative,

then all the elements of its inverse are positive.

MOSAK's and GOHEEN's proofs are founded on complete induction. In the present note we shall show that Theorem II can be proved directly by the use of the diadic representation of a matrix.[4])

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[1]) T. J. STIELTJES, Sur les racines de l'equation $X_n = 0$, *Acta Math.*, 9 (1886), 385—400.

[2]) J. L. MOSAK, *General equilibrium theory in international trade* (Cowles Commission Monograph, 1944), 49—51.

[3]) H. E. GOHEEN, On a lemma of Stieltjes on matrices, *Amer. Math. Monthly*, 56 (1949), 328—329.

[4]) See also E. EGERVÁRY, On a property of the projector matrices and its application to the canonical reduction of matrix functions, *these Acta*, 15 (1953), 1—6.

If $\mathbf{A}$ satisfies the condition II α then $a_{11}>0$ and we have the following identity:

$$\begin{vmatrix} a_{11} \cdots a_{1n} \\ \cdot \;\cdots\; \cdot \\ a_{n1} \cdots a_{nn} \end{vmatrix} - a_{11} \begin{vmatrix} 1 \\ a_{21}a_{11}^{-1} \\ \vdots \\ a_{n1}a_{11}^{-1} \end{vmatrix} \left| 1, \frac{a_{12}}{a_{11}}, \ldots, \frac{a_{1n}}{a_{11}} \right| =$$

$$= \frac{1}{a_{11}} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & \begin{vmatrix} a_{11}a_{12} \\ a_{21}a_{22} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11}a_{1n} \\ a_{21}a_{2n} \end{vmatrix} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \begin{vmatrix} a_{11}a_{12} \\ a_{n1}a_{n2} \end{vmatrix} & \cdots & \begin{vmatrix} a_{11}a_{1n} \\ a_{n1}a_{nn} \end{vmatrix} \end{bmatrix} = \frac{1}{a_{11}} \begin{vmatrix} 0 & 0 \\ 0 & \mathbf{A}' \end{vmatrix}.$$

It is easy to see that the matrix $\mathbf{A}'$ satisfies also the conditions of Theorem II. Indeed, the principal minors of $\mathbf{A}'$ can be deduced from those of $\mathbf{A}$ by multiplications and divisions[5]), and the elements off the main diagonal

$$a_{11}a_{ij} - a_{i1}a_{1j} \qquad (i \neq j, i \geqq 2, j \geqq 2)$$

are in virtue of the conditions II α, β obviously negative.

We introduce the notation

$$\begin{bmatrix} 1 \\ a_{21}a_{11}^{-1} \\ \vdots \\ a_{n1}a_{11}^{-1} \end{bmatrix} = \begin{bmatrix} 1 \\ -v_{21} \\ \vdots \\ -v_{n1} \end{bmatrix}; \quad \left[1, \frac{a_{12}}{a_{11}}, \ldots, \frac{a_{1n}}{a_{11}} \right] = [1, -w_{12}, \ldots, -w_{1n}]$$

and emphasize that the first elements of these vectors are positive and all remaining are negative.

Applying now the same process of reduction to $\mathbf{A}'$ and so on we arrive finally to the following diadic representation of $\mathbf{A}$[4])

$$\mathbf{A} = q_1 \begin{bmatrix} 1 \\ -v_{21} \\ -v_{31} \\ \vdots \\ -v_{n1} \end{bmatrix} [1, -w_{12}, -w_{13}, \ldots, -w_{1n}] + q_2 \begin{bmatrix} 0 \\ 1 \\ -v_{32} \\ \vdots \\ -v_{n2} \end{bmatrix} [0, 1, -w_{23}, \ldots, -w_{2n}] +$$

$$+ \cdots = \begin{bmatrix} 1 & 0 & \cdots 0 \\ -v_{21} & 1 & \cdots 0 \\ -v_{31} & -v_{32} & \cdots 0 \\ \vdots & \vdots & \vdots \\ -v_{n1} & -v_{n2} & \cdots 1 \end{bmatrix} \begin{bmatrix} q_1 0\, 0 \cdots 0 \\ 0\, q_2 0 \cdots 0 \\ 0\, 0\, q_3 \cdots 0 \\ \vdots \quad \vdots \\ 0\, 0\, 0 \cdots q_n \end{bmatrix} \begin{bmatrix} 1 & -w_{12} & -w_{13} & \cdots & -w_{1n} \\ 0 & 1 & -w_{23} & \cdots & -w_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} =$$

$$= (\mathbf{E}-\mathbf{M}) \langle q_1, \ldots, q_n \rangle (\mathbf{E}-\mathbf{N}),$$

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[5]) E. Pascal, *Die Determinanten* (Leipzig, 1900), 38—41.

where

$$q_1 = a_{11}; \quad q_k = \begin{vmatrix} a_{11} & & \\ & \ddots & \\ & & a_{kk} \end{vmatrix} : \begin{vmatrix} a_{11} & & \\ & \ddots & \\ & & a_{k-1,k-1} \end{vmatrix} \qquad (k = 2, 3, \dots, n)$$

and where all the elements q_k, r_{ij}, w_{ij} are positive.

Now we have

$$\mathbf{A}^{-1} = \{(\mathbf{E}-\mathbf{M})\langle q_1,\dots,q_n\rangle(\mathbf{E}-\mathbf{N})\}^{-1} = (\mathbf{E}-\mathbf{N})^{-1}\langle q_1,\dots,q_n\rangle^{-1}(\mathbf{E}-\mathbf{M})^{-1}.$$

But $\mathbf{M}$ and $\mathbf{N}$ are nilpotent matrices with positive elements and such that $\mathbf{M}^n = 0$, $\mathbf{N}^n = 0$, hence

$$(\mathbf{E}-\mathbf{M})^{-1} = \mathbf{E}+\mathbf{M}+\cdots+\mathbf{M}^{n-1},$$

$$\mathbf{A}^{-1} = (\mathbf{E}+\mathbf{N}+\cdots+\mathbf{N}^{n-1})\langle q_1^{-1}, q_2^{-1},\dots,q_n^{-1}\rangle(\mathbf{E}+\mathbf{M}+\cdots+\mathbf{M}^{n-1}).$$

The above expression clearly shows that $\mathbf{A}^{-1}$ is the product of two oppositely situated triangular matrices with positive elements and of a diagonal matrix with positive elements, consequently all the elements of $\mathbf{A}^{-1}$ are positive.

Some known results concerning the rigidity-matrix of a system of elastically connected particles suggest that the conditions of Theorem II can be replaced by weaker ones.

For example if the system is a string of n equal and equidistant particles with fastened ends, then the corresponding rigidity-matrix is[6])

$$(*) \qquad \mathbf{A}_n = \begin{bmatrix} 2 & -1 & 0 \cdots 0 \\ -1 & 2 & -1 \cdots 0 \\ 0 & -1 & 2 \cdots 0 \\ . & . & . \quad . \quad . \\ 0 & 0 & 0 \cdots 2 \end{bmatrix}, \quad |\mathbf{A}_n| = n+1,$$

and all the elements of its inverse[7])

$$\mathbf{A}_n^{-1} = \frac{1}{n+1}\begin{bmatrix} 1.n & 1(n-1)\cdots 1.2 & 1.1 \\ 1(n-1) & 2(n-1)\cdots 2.2 & 2.1 \\ 1(n-2) & 2(n-2)\cdots 3.2 & 3.1 \\ . & . \quad . \quad . \quad . & . \\ 1.1 & 2.1 \quad \cdots (n-1)1 & n.1 \end{bmatrix}$$

are positive. This result is however physically plausible, being a finite counterpiece to the wellknown theorem about the positivity of GREEN's function belonging to a continuous string.

As an extension of Theorem II to include such matrices as (*) we shall prove now the following

[6]) See f. i. E. J. ROUTH, *Advanced Dynamics*, Part II (1884), 226—228.

[7]) R. MISES—PH. FRANK, *Die Differential- und Integralgleichungen der Mechanik u. Physik* (Braunschweig, 1930) I. Teil, 502—503.

Theorem III. *Suppose that*

α) the principal minors of an n-th order matrix $\mathbf{A}_n$, *i. e.*

$$|\mathbf{A}_k| = \begin{vmatrix} a_{11}\, a_{12} \cdots a_{1k} \\ a_{21}\, a_{22} \cdots a_{2k} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{k1}\, a_{k2} \cdots a_{kk} \end{vmatrix} \qquad (k = 1, 2, \ldots, n)$$

are positive,

β) all the elements off its main diagonal are nonpositive,

γ) each column in the triangle above the diagonal and each row in the triangle under the diagonal contains at least one negative element.

Then all the elements of its inverse are positive.

Inasmuch

$$\mathbf{A}_2^{-1} = \begin{bmatrix} a_{11}\, a_{12} \\ a_{21}\, a_{22} \end{bmatrix}^{-1} = \begin{bmatrix} \dfrac{a_{22}}{|\mathbf{A}_2|} & -\dfrac{a_{12}}{|\mathbf{A}_2|} \\ -\dfrac{a_{21}}{|\mathbf{A}_2|} & \dfrac{a_{11}}{|\mathbf{A}_2|} \end{bmatrix},$$

the theorem is obviusly true for second order matrices satisfying the conditions III α, β, γ. Assume now that the theorem is true for any $n-1$-th order matrix satisfying the conditions III α, β, γ, and consider the following partitioned form of an n-th order matrix

$$\mathbf{A}_n = \left[\begin{array}{ccc:c} a_{11} & \cdots & a_{1,n-1} & a_{1n} \\ & & & \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & a_{n-1n} \\ \hdashline a_{n1} & \cdots & a_{n,n-1} & a_{nn} \end{array}\right] = \left|\begin{array}{c:c} \mathbf{A}_{n-1} & \mathbf{v}_n \\ \hdashline \mathbf{w}_n^* & a_{nn} \end{array}\right|$$

as well as its inverse[8])

$$\mathbf{A}_n^{-1} = \left[\begin{array}{c:c} \mathbf{A}_{n-1}^{-1} + \dfrac{|\mathbf{A}_{n-1}|}{|\mathbf{A}_n|}\mathbf{A}_{n-1}^{-1}\mathbf{v}_n\mathbf{w}_n^*\mathbf{A}_{n-1}^{-1} & -\dfrac{|\mathbf{A}_{n-1}|}{|\mathbf{A}_n|}\mathbf{A}_{n-1}^{-1}\mathbf{v}_n \\ \hdashline -\dfrac{|\mathbf{A}_{n-1}|}{|\mathbf{A}_n|}\mathbf{w}_n^*\mathbf{A}_{n-1}^{-1} & \dfrac{|\mathbf{A}_{n-1}|}{|\mathbf{A}_n|} \end{array}\right].$$

For sake of brevity let us introduce the notation $\mathbf{A} \succ 0$, if all the elements of $\mathbf{A}$ are positive. Obviously $\mathbf{A} \succ 0$, $\mathbf{B} \succ 0$ imply $\mathbf{A} + \mathbf{B} \succ 0$ and $\mathbf{A}\mathbf{B} \succ 0$.

We have by hypothesis

$$\mathbf{A}_{n-1}^{-1} \succ 0,$$

hence, by conditions β, γ,

$$-\mathbf{A}_{n-1}^{-1}\mathbf{v}_n \succ 0, \quad -\mathbf{w}_n^*\mathbf{A}_{n-1}^{-1} \succ 0$$

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8) See f. i. R. A. Fraser, W. J. Duncan and A. R. Collar, *Elementary Matrices* (Cambridge, 1938), 112—115.

and by condition α

$$\frac{|\mathbf{A}_{n-1}|}{|\mathbf{A}_n|} > 0,$$

consequently all the elements in each block of $\mathbf{A}_n^{-1}$ are positive. Q. e. d.

Note. After completion of the present paper the writer became acquainted with a paper of G. DE RHAM: Sur un théorème de Stieltjes relatif à certaines matrices, *Acad. Serbe des Sciences, Publications de l'Institut Math.*, **4** (1952), 133—134. Despite the fact that titles and topics are nearly identical, there is little overlap in content (only in the case of symmetrical matrices), and none in method.

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