On a theorem of Mautner.

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It is a few years ago that MAUTNER succeeded to prove an important result (cf. [3] Theorem 1) which asserts that every continuous unitary representation of a locally compact topological group in a Hilbert space can be decomposed into a continuous sum of irreducible continuous unitary representations. Roughly speaking this means that the space in which the given representation acts is represented as the direct sum of "virtual" subspaces, each "reducing" every operator of the representation, and such that, in each subspace, the "reduced" operators constitute a continuous irreducible representation of the group. This result generalizes the well-known fact, that a finite-dimensional space, in which a unitary representation is given, is the direct sum of mutually orthogonal minimal[1]) subspaces. In the formulation and proof of the above results fundamental role is played by the Reduction Theory of J. v. NEUMANN (cf. [5]).

In the course of a further discussion a natural question is how much the properties of the given representation are reflected in its decomposition. In particular, since the infinite dimensional representations in general have no minimal subspaces, it is of interest to obtain criteria in terms of the direct decomposition, which allow to conclude to the existence of such a subspace. In this connection the following theorem of MAUTNER is of importance (cf. [3] Theorem 3. 1):

Assumptions: Let $\mathfrak{H}$ be a separable Hilbert space and $\mathfrak{A}$ a family of bounded operators on $\mathfrak{H}$. Suppose that $\mathfrak{H}$ is the generalized direct sum of the Hilbert spaces $\mathfrak{H}_\lambda$ $(-\infty<\lambda<+\infty)$ with the weight function $\sigma(\lambda)$ (cf. [5] Definition 1) such that the operators of the family $\mathfrak{A}$ are decomposable, i. e. each $A\in\mathfrak{A}$ is the generalized direct sum of a system of operators $A(\lambda)$, in symbols $A\sim\Sigma A(\lambda)$ $(-\infty<\lambda<+\infty)$ (cf. [5] Definition 4). Since these decompositions $A(\lambda)$ for $A\in\mathfrak{A}$ are determined only up to a set of σ-measure 0, choose for

[1]) Given a family of operators $\mathfrak{A}$ in Hilbert space $\mathfrak{H}$, a subspace $\mathfrak{M}$ of $\mathfrak{H}$ is a minimal subspace with respect to $\mathfrak{A}$, if it is invariant under the operators of $\mathfrak{A}$, and has no proper subspaces enjoying the same property.

each $A \in \mathfrak{A}$ a representative $A(\lambda)$ from the class of operator valued functions $\bar{A}(\lambda)$ satisfying $A \sim \Sigma \bar{A}(\lambda)$, and suppose that, for each λ except those in a certain σ-null-set, $\mathfrak{H}_\lambda$ is irreducible under the family $\{A(\lambda)\}$. Suppose further that there exist at most countably many disjoint sets T_n $(n = 1, 2, \ldots)$ on the real line such that the complement of $\bigcup_n T_n$ is of σ-measure 0, and that for any λ and λ' in the same set T_n there exists a unitary operator $U(\lambda, \lambda')$ from $\mathfrak{H}_\lambda$ onto $\mathfrak{H}_{\lambda'}$ such that

$$A(\lambda') = U(\lambda, \lambda') A(\lambda) U^{-1}(\lambda, \lambda') \qquad \text{for all } A \in \mathfrak{A}.$$

Theorem A.[2]) *Under the above assumptions $\mathfrak{H}$ has proper minimal invariant subspaces with respect to the family $\mathfrak{A}$.*

Applying this theorem to unitary representations, it ensures the existence of a minimal invariant subspace under a representation if in its direct decomposition only countably many inequivalent irreducible representations occur (except for a set of σ-measure 0).

The purpose of the present note is to prove a theorem which generalizes Theorem A as far as we do not require the irreducibility of $\mathfrak{H}_\lambda$ under the family $\mathfrak{A}$. More precisely we prove the following

Theorem B. *Suppose that all assumptions of Theorem A are satisfied with the exception that the spaces $\mathfrak{H}_\lambda$ need not be irreducible under the families $\{A(\lambda)\}$ $(-\infty < \lambda < +\infty)$. Then there exists a subspace $\mathfrak{M}$ of $\mathfrak{H}$, invariant under the operators of $\mathfrak{A}$, and a unitary mapping U from $\mathfrak{M}$ onto $\mathfrak{H}_{\lambda_0}$ $(\lambda_0 \in T_n$, for a suitable $n)$, such that we have for all $A \in \mathfrak{A}$*

$$A(\lambda_0) = U A_{(\mathfrak{M})} U^{-1}$$

where $A_{(\mathfrak{M})}$ denotes the restriction of A in $\mathfrak{M}$.

Our proof follows closely the lines of the reasoning in [5], lemma 6 and lemma 7, p. 451.

Proof of Theorem B:

Since the complement of $\bigcup_n T_n$ is a σ-null-set, there exists necessarily an n for which T_n is not a σ-null set; we denote this set simply by T. Obviously the spaces $\mathfrak{H}_\lambda$ $(\lambda \in T)$ have all the same dimension. We assume that this dimension is ∞; the case when this dimension is a finite number may be treated quite similarly. We denote by Δ_∞ the set of those λ, for which $\dim \mathfrak{H}_\lambda = \infty$. Then Δ_∞ is σ-measurable ([5] Theorem 1) and $T \subset \Delta_\infty$.

Choose a measurable family $\varphi_k(\lambda)$ $(k = 1, 2, \ldots)$ (cf. [5] Definition 2), a Hilbert space $\mathfrak{H}_0$, a complete orthonormal system ψ_k $(k = 1, 2, \ldots)$ in it, and define an isomorphism J_λ between $\mathfrak{H}_\lambda$ $(\lambda \in \Delta_\infty)$ and $\mathfrak{H}_0$ under which $\varphi_k(\lambda)$

[2]) The proof given in [3] is incomplete, because of the lacking justification for the assertion, that the closure of an algebraic operator-ring in the strong topology coincides with its strong sequential closure. But as Mr. J. Dixmier kindly informed me, this follows in the case of a separable Hilbert space from Theorem 1 in [1], as it is easily seen.

corresponds to ψ_k $(k=1,2,\ldots)$. Let λ_0 be a point of T. Applying now a theorem of J. v. NEUMANN (cf. [4] p. 386) we can select a countable subfamily $\mathfrak{A}_0$ from $\mathfrak{A}$ such that $\mathfrak{A}_0$ and $\{A(\lambda_0)\}$ $(A\in\mathfrak{A}_0)$ are dense in $\mathfrak{A}$ and $\{A(\lambda_0)\}$ $(A\in\mathfrak{A})$, respectively, in the strong sequential topology. Let A_k $(k=1,2,\ldots)$ be the elements of $\mathfrak{A}_0$ with the decompositions $A_k\sim\Sigma A_k(\lambda_0)$ $(-\infty<\lambda<+\infty)$. Put

$$\widetilde{A}_k(\lambda)=\begin{cases} J_\lambda A_k(\lambda)J_\lambda^{-1} & \text{for } \lambda\in\Delta_\infty \\ O & \text{for } \lambda\notin\Delta_\infty \end{cases} \quad (k=1,2,\ldots).$$

Form as in [5], lemma 6, the product space $r\times S$, where r is the set of real numbers, S the space of all linear transformations in $\mathfrak{H}_0$ with a norm $\leqq 1$ in their weak topology. Then $r\times S$ is a complete separable metric space (cf. [5], pp. 447—448). Consider now the set B of pairs (λ, V) $(-\infty<\lambda<\infty;$ $V\in S)$ in $r\times S$ satisfying the following equations:

$$(*)\qquad \left.\begin{cases} (\widetilde{A}_k(\lambda)V\psi_\nu, V\psi_\mu)=(\widetilde{A}_k(\lambda_0)\psi_\nu, \psi_\mu) \\ (V\psi_\nu, V\psi_\mu)=\delta_{\nu\mu} \\ (V^*\psi_\nu, V^*\psi_\mu)=\delta_{\nu\mu} \end{cases}\right\} (k,\nu,\mu=1,2,\ldots).$$

We show now similarly as it is done in [5], lemma 6, that B is a Borel set in the space $r\times S$, at least if the $\widetilde{A}_k(\lambda)$ are previously modified on a suitable set of σ-measure 0, independent of k. Since

$$(\widetilde{A}_k(\lambda)V\psi_\nu, V\psi_\mu)=\Sigma_{\varrho,\tau}(V\psi_\nu,\psi_\varrho)(\widetilde{A}_k(\lambda)\psi_\varrho,\psi_\tau)\overline{(V\psi_\mu,\psi_\tau)},$$
$$(V\psi_\nu, V\psi_\mu)=\Sigma_\tau(V\psi_\nu,\psi_\tau)\overline{(V\psi_\mu,\psi_\tau)},$$
$$(V^*\psi_\nu, V^*\psi_\mu)=\Sigma_\tau(\psi_\nu, V\psi_\tau)\overline{(\psi_\mu, V\psi_\tau)},$$

we need only to make the functions $f_{k,\mu,\nu}(\lambda)=(\widetilde{A}_k(\lambda)\psi_\mu,\psi_\nu)$ Borel measurable, because the functions $(V\psi_\nu,\psi_\mu)$ (of V) are clearly continuous functions in $r\times S$. But since the set of these functions is countable we can find a set $N\subset\Delta_\infty$ of σ-measure 0, such that if we put $\tilde{\tilde{A}}_k(\lambda)=\widetilde{A}_k(\lambda)$ on $\Delta_\infty-N$ and $\tilde{\tilde{A}}_k(\lambda)=0$ on N $(k=1,2,\ldots)$, then the functions $(\tilde{\tilde{A}}_k(\lambda)\psi_\mu,\psi_\nu)$ become Borel measurable (in the following we write again $\widetilde{A}_k(\lambda)$ instead of $\tilde{\tilde{A}}_k(\lambda)$).

Now we apply lemma 5 of [5]. From this it follows that the set K of the λ's for which there exists a $V\in S$ such that $(\lambda, V)\in B$, is σ-measurable, and that there exists a mapping $\lambda\to(\lambda, V(\lambda))$ from K to B such that the inverse image of every open set O in $r\times S$ is also σ-measurable.

We put

$$U(\lambda)=\begin{cases} J_\lambda^{-1}V(\lambda)J_\lambda & \text{if } \lambda\in K\cap\Delta_\infty, \\ O & \text{if } \lambda\notin K\cap\Delta_\infty. \end{cases}$$

If $\lambda\in K$, then $U(\lambda)$ is unitary and depends σ-measurably on λ (cf. [5] Definition 5). Since the proof of the latter statement requires essentially only the repetition of the argument in [5], p. 453, we omit the further discussion.

Observe now that by the definition of the set T, for each $\lambda \in \dot{T}$ there exists a $V \in S$ such that (λ, V) satisfies the equations (*). Since in the course of our construction the operator-valued functions $\tilde{A}_k(\lambda)$ are modified on a set of σ-measure 0 only, the set $T-K$ is a σ-null set. Therefore, by our assumption on T, K is necessarily of positive σ-measure. Put[3])

$$\underline{\omega}_j(\lambda) = U(\lambda)\underline{\varphi}_j(\lambda) \qquad (-\infty < \lambda < +\infty)$$

and

$$\omega_j = \frac{1}{\sqrt{\sigma(K)}} \int_K \underline{\omega}_j(\lambda) \sqrt{d\sigma(\lambda)} \qquad (j = 1, 2, \ldots)$$

(cf. [5] Definition 1).

The system $\{\omega_k\}$ $(k = 1, 2, \ldots)$ is orthonormal in $\mathfrak{H}$, and the correspondence $\omega_k \to \underline{\varphi}_k(\lambda_0)$ defines an isomorphism U between the spaces $\mathfrak{M}$ spanned by the ω_k and $\mathfrak{H}_{\lambda_0}$. Putting

$$a_{j,l}^{(k)} = (A_k(\lambda_0)\underline{\varphi}_j(\lambda_0), \underline{\varphi}_l(\lambda_0)) \qquad (k, j, l = 1, 2, \ldots)$$

we get

$$A_k\omega_j = \frac{1}{\sqrt{\sigma(K)}} \int_K A_k(\lambda)\underline{\omega}_j(\lambda)\sqrt{d\sigma(\lambda)} =$$

$$= \sum_{l=1}^{\infty} \frac{1}{\sqrt{\sigma(K)}} \int_K (A_k(\lambda)\underline{\omega}_j(\lambda), \underline{\omega}_l(\lambda))\underline{\omega}_l(\lambda)\sqrt{d\sigma(\lambda)} = \sum_{l=1}^{\infty} a_{jl}^{(k)}\omega_l.$$

This equation shows that $\mathfrak{M}$ is invariant under the family $\mathfrak{A}_0$, and that the above isomorphism carries the restriction of $A \in \mathfrak{A}_0$ in $\mathfrak{M}$ into the operator $A(\lambda_0)$. But by our choice of $\mathfrak{A}_0$ this clearly extends to all operators of $\mathfrak{A}$, i. e. we have

$$A(\lambda_0) = UA_{(\mathfrak{M})}U^{-1} \qquad (A \in \mathfrak{A}).$$

This proves the theorem.

Bibliography.

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[3]) For a similar reasoning cf. [2], p. 12.