The theorem of Radon—Nikodym in operator-rings.

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Preliminaries. In the following $\mathbf{M}$ denotes a weakly closed ring of operators on a Hilbert space $\mathfrak{H}$, which contains no purely infinite projections, i. e. we suppose that for every projection $P \in \mathbf{M}$ there exists a finite projection[1]) $Q \in \mathbf{M}$ with $Q \leqq P$. As it is shown in [1] (especially ch. V., p. 40) there exists a suitable two-sided ideal $\mathfrak{m}$ of $\mathbf{M}$, and a *positive* linear functional φ defined on $\mathfrak{m}$, with the following additional properties:

(i) For every $B \in \mathbf{M}$ and $A \in \mathfrak{m}$ we have $\varphi(AB) = \varphi(BA)$.

(ii) If $A \in \mathfrak{m}$ is the l. u. b. of an increasing directed set F of positive operators im $\mathfrak{m}$, then we have $\varphi(A) = \underset{A' \in F}{\text{l. u. b.}}\ \varphi(A')$.

(iii) If $\varphi(P) = 0$ for a projection $P \in \mathfrak{m}$, it follows that $P = O$.

(iv) The l. u. b. of the projections[2]) in $\mathfrak{m}$ is the unit operator I.

(v) φ is maximal i. e. it cannot be extended to a two-sided ideal of $\mathbf{M}$ which contains $\mathfrak{m}$ properly.

We shall call such a functional φ a *trace*. Property (v) implies that if $A \in \mathbf{M}$ is the l. u. b. of an increasing directed set F of positive operators in $\mathfrak{m}$ such that the $\underset{A' \in F}{\text{l. u. b.}}\ \varphi(A')$ is finite, then $A \in \mathfrak{m}$, and hence by property (ii) $\varphi(A) = \underset{A' \in F}{\text{l. u. b.}}\ \varphi(A')$ (cf [1] lemma 5. 5). The two-sided ideal generated by the projections of $\mathfrak{m}$ will be denoted by $\mathfrak{m}^r$. For every operator $A \in \mathfrak{m}^r$ there exists a projection P in $\mathfrak{m}$ such that $PA = AP = A$, and for every $A \in \mathbf{M}$, $A \geqq O$, the set of positive operators in m^r bounded by A forms an increasing directed set with an upper bound equal to A.

In particular if $\mathbf{M}$ is *finite* (i. e. contains only finite projections) and *σ-finite* (see below) then we can take $\mathfrak{m} = \mathbf{M}$. We shall use also the fact that the restriction of φ to a projection $P \in \mathfrak{m}$, i. e. the functional $\psi(A) = \varphi(PA)$ is continuous in the weak sequential topology [cf. [2], Theorem 3, Corollary 8].

By ϱ and τ we shall denote two positive functionals on M, which are continuous in the strong and hence also in the weak topology if they are

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[1]) A projection P in an operator-ring $\mathbf{M}$ is called *finite* if $\mathbf{M}$ contains no partial isometry with initial domain $P\mathfrak{H}$ and terminal domain $Q\mathfrak{H} \subset P\mathfrak{H}$, $Q\mathfrak{H} \neq P\mathfrak{H}$.

[2]) The l. u. b. of a set $\{P_\alpha\}$ of projections in $\mathbf{M}$ is the minimal projection $P \in \mathbf{M}$ such that $P_\alpha \leqq P$ for all $\{P_\alpha\}$.

restricted to the unit sphere [cf. [2], Theorem 2]. ϱ will be supposed *absolutely continuous* with respect to τ, in symbols $\varrho \prec \tau$, in the sense that if $\tau(P)=0$ for a projection $P \in \mathbf{M}$, than we have also $\varrho(P)=0$.

A projection $P \in \mathbf{M}$ is called σ-finite, if every disjoint family of projections in $\mathbf{M}$ bounded by P is at most countable. The ring $\mathbf{M}$ itself is said to be σ-finite if the operator I is σ-finite.

We denote by $\mathfrak{m}_p$ (resp. $\mathbf{M}_p$) the set of projections in $\mathfrak{m}$ (resp. in $\mathbf{M}$).

A closed, densely defined operator T is said to „belong" to $\mathbf{M}$, in symbols $T \eta \mathbf{M}$, if it commutes with every operator in $\mathbf{M}'$.

Introduction. In a recent paper [7] I. E. SEGAL has given a systematic development of a theory of „non-commutative integration" in operator-rings which summarizes in a certain sense the various analogies between the theory of measures and the theory of rings of operators emphasized already by F. MURRAY and J. VON NEUMANN in their fundamental papers and developed by several authors in these years. In particular SEGAL generalizes by aid of this theory a special case of the important extension of the classical Radon—Nikodym theorem by H. A. DYE (cf. [7], Theorem 14, p. 433; and [3], Corollary 5. 1 and Theorem 4, p. 268).

The purpose of the present note is to give an alternative approach to both theorems; instead of SEGAL's theory of integration we use J. DIXMIER's theory of traces in arbitrary operator-rings, cf. [1]. Firstly we prove a theorem (Theorem I) which is equivalent to the theorem of SEGAL cited above, and then, in the case of a finite and σ-finite ring, we derive from this DYE's extension of the Radon—Nikodym theorem (Theorem II). We formulate these theorems as follows:

Theorem I. *Let φ be a trace defined on the two-sided ideal $\mathfrak{m}$ of the ring $\mathbf{M}$. Then to every positive functional ϱ on $\mathbf{M}$ we can find a positive (in general unbounded) operator $H=\int_0^\infty \lambda\, dE_\lambda$, $H \eta \mathbf{M}$, such that for every $A \in \mathbf{M}$ we have*

$$\varrho(A)=\varphi(AH).$$

If H is unbounded, the right-hand side of the last equation is interpreted as $\lim_{\substack{\mu \to \infty \\ \varepsilon \to 0}} \varphi(AH_\varepsilon^\mu)$, where

$$H_\varepsilon^\mu=\int_\varepsilon^\mu \lambda\, dE_\lambda \in \mathfrak{m}.$$

Theorem II. *If $\mathbf{M}$ is finite and σ-finite, and ϱ and τ are two positive functionals such that $\varrho \prec \tau$ then there exists a closed, densely defined operator T, $T \eta \mathbf{M}$, such that*

$$\varrho(A)=\tau(T^* A T) \qquad \textit{for all } A \in \mathbf{M}.$$

If T is unbounded, then, $T = WH$ being the polar decomposition of T and $H = \int_0^\infty \lambda dE_\lambda$, $\tau(T^* A T)$ is interpreted as $\lim_{n\to\infty} \tau(T_n^* A T_n)$, where $T_n = TE_n$, $T_n \in \mathbf{M}$.

Our proof of I consists in a direct construction of H by a method which became already classical in the theory of semi-ordered spaces and its applications, especially in the spectral theory and theory of measures cf. [6]. Important role is played in the course of the proof by a lemma which is similar to one used by MURRAY and v. NEUMANN in [5] (cf. lemma II).

1. Firstly we need some lemmas.

Lemma I. (For a similar reasoning cf. [5], lemma 3. 2. 1 and 3. 2. 2.) *φ and ϱ being defined as in Theorem I form the functional $\varrho_\lambda = \varrho - \lambda\varphi$ defined on $\mathfrak{m}$, for $\lambda > 0$, and put $m_\lambda = \text{l. u. b.}_{P \in \mathfrak{m}_p} \varrho_\lambda(P)$ ($\leqq \varrho(I)$). Then there exists a projection $E_\lambda \in \mathfrak{m}_p$ such that $\varrho_\lambda(E_\lambda) = m_\lambda$.*

Proof: Put $n_\lambda = \text{l. u. b.}_{\substack{O \leqq A \leqq I \\ A \in \mathfrak{m}}} \varrho_\lambda(A)$; we show first the existence of an operator $A_0 \in \mathfrak{m}$ with $\varrho_\lambda(A_0) = n_\lambda$. To do this, we choose a sequence of operators $A_n \in \mathfrak{m}$ ($n = 1, 2, \ldots$) such that $O \leqq A_n \leqq I$ and $\lim_{n\to\infty} \varrho_\lambda(A_n) = n_\lambda$. Observe that by the properties of ϱ, φ and $\mathfrak{m}^r$ we can choose $A_n \in \mathfrak{m}^r$; from this follows the existence of a sequence of projections $P_n \in \mathfrak{m}$ ($n = 1, 2, \ldots$) such that $P_n A_n = A_n$ ($n = 1, 2, \ldots$). Each P_n is σ-finite, because $\varphi(P_n)$ is finite and since $\varphi(P) > 0$ for $P \neq O$. Therefore if we put $P = \text{l. u. b.}_{n=1,2,\ldots} P_n$, P is the l. u. b. of a sequence of projections of the form $P_{z_\nu}^{\mathbf{M}'}$ [3]) ($\nu = 1, 2, \ldots$). Using the diagonal process we can determine a subsequence of the sequence A_n (we denote it again by A_n) such that for every fixed ν the sequence $A_n z_\nu$ converges weakly in $\mathfrak{H}$. But since the elements of the form $\sum_{\mu=1}^m A'_\mu z_\mu$, $A'_\mu \in \mathbf{M}'$, are dense in $P\mathfrak{H}$, the same is true for any sequence $A_n y$ with $y \in P\mathfrak{H}$ and hence for every sequence $A_n x$ with x arbitrary in $\mathfrak{H}$. Putting

$$\text{weak}\lim_{n\to\infty} A_n x = A_0 x \quad \text{for} \quad x \in \mathfrak{H}$$

we have evidently $A_0 \in \mathbf{M}$, $O \leqq A_0 \leqq I$ and, by definition, $A_0 = \text{weak}\lim_{n\to\infty} A_n$.

Next we show that $\varrho_\lambda(A_0) = n_\lambda$. Observe that by the definition of the A_n's, and by the continuity property of ϱ (cf. Preliminaries) $\lim_{n\to\infty} \varphi(A_n) = \alpha$ exists, and hence it suffices to show that $A_0 \in \mathfrak{m}$ and $\varphi(A_0) = \alpha$. But by the maximal property of φ this is contained in the following statement: if $O \leqq B \in \mathfrak{m}^r$ and $B \leqq A_0$ then $\varphi(B) \leqq \alpha$ (cf. Preliminaries). Indeed $B \in \mathfrak{m}^r$ implies

[3]) $P_z^{\mathbf{M}'}$ denotes the projection on the subspace spanned by the vectors $A'z$ ($A' \in \mathbf{M}'$).

$PB=B$ for some $P\in \mathfrak{m}_p$, hence $B\leqq PA_0P\in \mathfrak{m}$ and $0\leqq \varphi(B)\leqq \varphi(PA_0P)= =\lim_{n\to\infty}\varphi(PA_nP)\leqq \lim_{n\to\infty}\varphi(A_n)=\alpha$, φ being weak-sequential continuous if restricted to finite projections.

Finally we show that we may replace A_0 by a projection P_0 enjoying the same maximum property as A_0; P_0 will have then evidently the properties required for E_λ. Let $A_0=\int_0^1 \mu dF_\mu$ be the spectral representation of A_0. Then we have $\varrho_\lambda(A_0F_a)=\int_0^a \mu d(-\varrho_\lambda(I-F_\mu))$ [4]) for every $a\geqq 0$. $\varrho_\lambda(A_0F_a)$ is an increasing function of a; otherwise we could find an interval $\Delta=[\alpha,\beta]$ such that putting $F(\Delta)=F_\beta-F_\alpha$ we should have $\varrho_\lambda(F(\Delta)A_0)<0$, and this would imply $\varrho_\lambda((I-F(\Delta))A_0)>\varrho_\lambda(A_0)$, but this contradicts to the definition of A_0, because plainly $A_0(I-F(\Delta))\in\mathfrak{m}$ and $O\leqq A_0(I-F(\Delta))\leqq I$.

Thus $-\varrho_\lambda(I-F_\mu)$ is an increasing function of μ if $\mu>0$; hence, using also the maximal property of A_0, we have

$$\int_a^1 \mu d(-\varrho_\lambda(I-F_\mu))\leqq \varrho_\lambda(I-F_a)\leqq \varrho_\lambda(A_0)=n_\lambda$$

for every $a>0$. For $a\to 0$ the integral tends to $\varrho_\lambda(A_0)$, thus (noting the definition of ϱ_λ) $I-F_{+0}\in\mathfrak{m}$ and $\varrho_\lambda(I-F_{+0})=n_\lambda$, i. e. $n_\lambda=m_\lambda$.

Note that we may suppose that for any $P\leqq E_\lambda$ with $\varrho_\lambda(P)=0$ we have $P=O$, for otherwise choose a maximal system of orthogonal projections $\{P_\mu\}\leqq E_\lambda$ such that $\varrho_\lambda(P_\mu)=0$. By the countable additivity of ϱ_λ for projections $\leqq E_\lambda$, $E_\lambda-\sum_{\mu=1}^{\infty}P_\mu$ in place of E_λ evidently has the required property.

Lemma II. (Cf. [4], lemma 3. 3. 1.) *Let E_λ as defined in lemma I. Then for every $A\in\mathbf{M}$ we have $\varrho(AE_\lambda)=\varrho(E_\lambda A)$.*

Proof: Evidently it suffices to prove this for ϱ_λ in place of ϱ, and for a self-adjoint $A\in\mathbf{M}$. Consider the function: $\varphi_\varepsilon(\alpha)=\dfrac{1+i\varepsilon\alpha}{1-i\varepsilon\alpha}$ $(\varepsilon\gtrless 0)$. As $|\varphi_\varepsilon(\alpha)|\equiv 1$, $U=\varphi_\varepsilon(A)$ is $\in\mathbf{M}$ and unitary. We have $\varphi_\varepsilon(\alpha)=1+2i\varepsilon\alpha+\varepsilon^2\psi_\varepsilon(\alpha)$ where $\psi_\varepsilon(\alpha)=\dfrac{-2\alpha^2}{1-i\varepsilon\alpha}$, further $\|\psi_\varepsilon(A)\|\leqq 2\|A\|^2$. By the maximal property of E_λ we have

$$\begin{aligned}0&\leqq \varrho_\lambda(E_\lambda)-\varrho_\lambda(UE_\lambda U^*)=\\&=\varrho_\lambda(E_\lambda)-\varrho_\lambda([I+2i\varepsilon A+\varepsilon^2\psi_\varepsilon(A)]E_\lambda[I-2i\varepsilon A+\varepsilon^2\overline{\psi}_\varepsilon(A)])=\\&=2i\varepsilon\varrho_\lambda(AE_\lambda-E_\lambda A)+O(\varepsilon^2).\end{aligned}$$

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[4]) Observe that $I-F_{\mu_0}\in\mathfrak{m}$ for $\mu_0>0$ if $A_0\in\mathfrak{m}$ because $I-F_{\mu_0}=A_0T$, where $T=\int_{\mu_0}^1 \mu^{-1}dF_\mu\in\mathbf{M}$.

But this implies the result announced, because ε is arbitrary $\gtrsim 0$.

R e m a r k. We note that by a similar reasoning as applied in lemma I and II we can obtain the following result: *Let ϱ be a real linear functional (i. e. which takes real values for selfadjoint operators) defined on an arbitrary operator-ring* $\mathbf{M}$ *and continuous in the strong* (hence also in the weak) *topology if restricted to the unit sphere of* $\mathbf{M}$ (cf. [2] Theorem 2). *Then ϱ can be represented as the difference of two disjoint positive functionals defined on* $\mathbf{M}$. (Here we call two positive functionals disjoint, if their carrier projections are orthogonal; cf. [3] p. 264). To see this we denote by S the positive part of the unit sphere of $\mathbf{M}$, and determine an operator $A \in S$ such that $\varrho(A) = \sup_{A' \in S} \varrho(A')$. For this it suffices to remark that S is compact in the weak topology and hence the continuous real-valued function $\varrho(A')$ $(A' \in S)$ takes its maximum for an $A \in S$. Next applying the corresponding reasonings of the lemmas I and II (replacing ϱ_λ by ϱ) we can show that A can be taken as a projection $E \in \mathbf{M}$ and that $\varrho(EA) = \varrho(AE)$ for every $A \in \mathbf{M}$. Finally putting $\varrho_1(A) = \varrho(EA)$, $\varrho_2(A) = -\varrho((I-E)A)$ $(A \in \mathbf{M})$, $\varrho = \varrho_1 - \varrho_2$ plainly yields the desired representation.

L e m m a III. *For $\lambda \geqq \mu$ we have $E_\lambda \leqq E_\mu$.*

P r o o f: It follows from the definition of E_λ that if $P \in \mathbf{M}, P \leqq E_\lambda$, then we have $\varrho_\lambda(P) \geqq 0$; indeed, $PE_\lambda = P$ gives $P \in \mathfrak{m}_\mu$ and if $\varrho_\lambda(P) < 0$ then $\varrho_\lambda(E_\lambda - P) > \varrho_\lambda(E_\lambda)$ which contradicts to the definition of E_λ. Hence we have also for every positive $A \in \mathbf{M}$ with $E_\lambda A = A$: $\varrho_\lambda(A) \geqq 0$. Applying lemma II it follows $\varrho_\lambda(E_\lambda(I-E_\mu)) = \varrho_\lambda(E_\lambda(I-E_\mu)E_\lambda) \geqq 0$. Similarly if $A(I-E_\mu) = A$ and $A \in \mathfrak{m}, A \geqq O$, we have $\varrho_\mu(A) \leqq 0$ and since $\varrho_\mu(A) \geqq \varrho_\lambda(A)$ for every $A \geqq O$, it follows (again by lemma II)

$$\varrho_\lambda(E_\lambda(I-E_\mu)) = \varrho_\lambda([I-E_\mu]E_\lambda[I-E_\mu]) \leqq \varrho_\mu([I-E_\mu]E_\lambda[I-E_\mu]) \leqq 0.$$

But these inequalities together imply $\varrho_\lambda(E_\lambda(I-E_\mu)) = 0$. From this it follows by the remark at the end of lemma I that $E_\lambda(I-E_\mu)E_\lambda = O$ or $(I-E_\mu)E_\lambda = O$, $E_\lambda = E_\mu E_\lambda$, qu. e. d.

P r o o f of T h e o r e m I: By lemma III $\lim_{\lambda \to \infty} E_\lambda = E$ exists. The reasoning in lemma III shows that $\varrho_\lambda(E) \geqq 0$ for every $\lambda > 0$, hence $0 \leqq \varphi(E) \leqq \lim_{\lambda \to \infty} \frac{\varrho(I)}{\lambda} = 0$, which gives $E = O$. Similarly we can put $F = \lim_{\varepsilon \to 0} E_\varepsilon$. If $Q \in \mathfrak{m}_\mu$ and $Q \leqq I-F$ we have by the above argument $\varrho_\varepsilon(Q) \leqq 0$ for every $\varepsilon > 0$, hence $\varrho(Q) = 0$. But since $I-F$ is the l. u. b. of the projections $Q \in \mathfrak{m}_\mu$, $Q \leqq I-F$ (cf. Preliminaries; this follows from property (iv) of φ), by the continuity property of ϱ we have $\varrho(I-F) = 0$.

We define now

$$F_\lambda = \begin{cases} O & \text{if } \lambda \leqq 0, \\ I-E_\lambda & \text{if } \lambda > 0; \end{cases} \qquad H = \int_0^\infty \lambda \, dF_\lambda,$$

and show that H satisfies the requirements of Theorem I. For this it suffices to prove that

$$\varrho(AF_\varepsilon^\mu)=\varphi(AH_\varepsilon^\mu) \quad \text{(we put } F_\varepsilon^\mu=F_\mu-F_\varepsilon)$$

for every $A\in\mathbf{M}$ and $\mu>\varepsilon>0$, because by the strong sequential continuity of ϱ and by the definition of F_λ we have $\lim\limits_{\substack{\varepsilon\to 0\\ \mu\to\infty}}\varrho(AF_\varepsilon^\mu)=\varrho(A)$.

Consider now a subdivision of the interval $[\varepsilon,\mu]$ by the points $\varepsilon=\lambda_0<\lambda_1<\ldots<\lambda_\mu=\mu$. Let Δ_k be the interval $(\lambda_{k-1},\lambda_k)$. For every projection P with $P\leqq F(\Delta_k)=F_{k_\lambda}-F_{\lambda_{k-1}}=E_{\lambda_{k-1}}-E_{\lambda_k}$ we have $0\leqq\varrho_{\lambda_{k-1}}(P)$, and hence for every $A\geqq O$ with $AF(\Delta_k)=A$ we have

$$0\leqq\varrho_{\lambda_{k-1}}(A)=\varrho(A)-\lambda_{k-1}\varphi(A)\leqq(\lambda_k-\lambda_{k-1})\varphi(A).$$

If $A\in\mathbf{M}$ we have by lemma II:

$$|\varrho(AF_\varepsilon^\mu)-\textstyle\sum_{k=1}^n\lambda_{k-1}\varphi(AF(\Delta_k))|\leqq$$
$$\textstyle\sum_{k=1}^n|\varrho(F(\Delta_k)AF(\Delta_k))-\lambda_{k-1}\varphi(F(\Delta_k)AF(\Delta_k))|\leqq\|A\|\sup\limits_{k=1,2,\ldots}(\lambda_k-\lambda_{k-1})\varphi(F_\varepsilon^\mu).$$

Hence finally, putting $H(\Delta_k)=\int\limits_{\lambda_{k-1}}^{\lambda_k}\lambda dF_\lambda$, we have for an arbitrary $A\in\mathbf{M}$

$$|\varrho(AF_\varepsilon^\mu)-\varphi(AH_\varepsilon^\mu)|\leqq\textstyle\sum_{k=1}^n|\varrho(AF(\Delta_k))-\lambda_{k-1}\varphi(AF(\Delta_k))|+$$
$$+\textstyle\sum_{k=1}^\mu|\varphi(AH(\Delta_k))-\lambda_{k-1}\varphi(AF(\Delta_k))|\leqq 2\|A\|\sup\limits_{k=1,2,\ldots}(\lambda_k-\lambda_{k-1})\varphi(F_\varepsilon^\mu).$$

Since $\sup\limits_{k=1,2,\ldots}(\lambda_k-\lambda_{k-1})$ is arbitrary small the desired equality follows.

2. In the following we shall suppose that $\mathbf{M}$ is a finite and σ-finite ring, ϱ and τ are positive functionals on $\mathbf{M}$ which are strongly continuous on the unit sphere, $\varrho\prec\tau$ and the trace φ is defined for every $A\in\mathbf{M}$, or $\mathfrak{m}=\mathbf{M}$ (for these cf. Preliminaries).

Before passing to the proof of theorem II, we need two lemmas[5]):

Lemma IV. *Suppose that $\varrho(A)\leqq(Ax,x)$ for a fixed element $x\in\mathfrak{H}$ and for every positive $A\in\mathbf{M}$. Then there exists an $y\in\mathfrak{H}$ such that $\varrho(A)=(Ay,y)$ for every $A\in\mathbf{M}$.*

Proof: Consider the subspace $\mathfrak{M}$ of $\mathfrak{H}$ spanned by the elements $\{Tx\}$, $T\in\mathbf{M}$. For $u=Tx$ and $v=Sx$ $(T,S\in\mathbf{M})$ the number $(u,v)_1=\varrho(S^*T)$ depends only on u and v, because by assumption from $Tx=0$ $(T\in\mathbf{M})$ follows

$$0\leqq(Tx,Tx)_1=\varrho(T^*T)\leqq(Tx,Tx)=0 \quad\text{or}\quad (Tx,Tx)_1=0.$$

Hence $(u,v)_1$ defines a bilinear form for the elements $\{Tx\}$ $(T\in\mathbf{M})$ of $\mathfrak{M}$, which is evidently bounded and positive. Therefore it can be extended by continuity to all elements of $\mathfrak{M}$. By a familiar theorem of the theory of oper-

[5]) These are essentially Lemma 2.2 and Theorem 1 in [3]. We repeat the slightly modified proofs for the convenience of the reader.

ators there exists a positive operator B on $\mathfrak{M}$ such that $(u,v)_1=(Bu,v)$ for every $u,v\in\mathfrak{M}$. Putting $u=Tx, v=Sx, (S,T\in\mathbf{M})$ we have $(BAu,v)= =(Au,v)_1=\varrho(S^*AT)=(u,A^*v)_1=(ABu,v)$ for every $S,T,A\in\mathbf{M}$ which gives $AB=BA$ on $\mathfrak{M}$ for a fixed A. Hence, finally, putting $y=B^{\frac{1}{2}}x$:

$$\varrho(A)=(BAx,x)=(AB^{\frac{1}{2}}x,B^{\frac{1}{2}}x)=(Ay,y) \qquad (A\in\mathbf{M}).$$

Lemma V. *For every $P\in\mathbf{M}_p$ there exists a $Q\in\mathbf{M}_p$, $Q\leqq P$, and an element $y\in\mathfrak{H}$ such that $\varrho(QAQ)=(Ay,y)$ for $A\in\mathbf{M}$.*

Proof: We need only to show that every $P\in\mathbf{M}_p$ contains a $Q\in\mathbf{M}_p$ such that for every $Q'\leqq Q, Q'\in\mathbf{M}_p$, we have $\varrho(Q')\leqq(Q'x,x)$ with a fixed $x\in\mathfrak{H}$. Indeed, from this it follows evidently that $\varrho(A)\leqq(Ax,x)$ for every positive $A\in\mathbf{M}$ such that $QA=A$; but by lemma IV this implies $\varrho(QAQ)=(Ay,y)$ for $A\in\mathbf{M}$ and a suitable $y\in\mathfrak{H}$.

Choose now $x\in P\mathfrak{H}$ such that $\varrho(P)\leqq(x,x)$. If we have $\varrho(Q')\leqq(Q'x,x)$ for $Q'\leqq P, Q'\in\mathbf{M}_p$, then $P=Q$ and x satisfies our requirements. Otherwise choose a maximal system of orthogonal projections $\{P_\alpha\}\in\mathbf{M}_p$ $(\alpha=1,2,\ldots)$ bounded by P, and such that $\varrho(P_\alpha)>(P_\alpha x,x)$ $(\alpha=1,2,\ldots)$. We have necessarily $P-\sum_{\alpha=1}^{\infty}P_\alpha\neq O$; for otherwise using the continuity property of ϱ we should have $\varrho(P)>(x,x)$, which contradicts to the choice of x. Hence, putting $Q=P-\sum_{\alpha=1}^{\infty}P_\alpha$, Q evidently meets our requirements.

Proof of Theorem II: Replace ϱ by φ in lemma V and let $\{P_\alpha\}$[6] $(\alpha=1,2\ldots)$ be a maximal system of orthogonal projections $\in\mathbf{M}$ enjoying the property described there for Q: for every α $(\alpha=1,2,\ldots)$ there exists an element $x_\alpha\in\mathfrak{H}$ such that $\varphi(P_\alpha AP_\alpha)=\varphi(P_\alpha A)=(Ax_\alpha,x_\alpha)$ for $A\in\mathbf{M}$. By lemma V we have necessarily $\sum_{\alpha=1}^{\infty}P_\alpha=I$. Using the strong continuity of φ it follows

$$\varphi(A)=\sum_{\alpha=1}^{\infty}\varphi(AP_\alpha)=\sum_{\alpha=1}^{\infty}(Ax_\alpha,x_\alpha).$$

Applying Theorem I, the positive functionals ϱ and τ can be represented for all $A\in\mathbf{M}$ in the following form: $\varrho(A)=\varphi(AH'')$, $\tau(A)=\varphi(AH')$, where H',H'' are positive hermitian operators belonging to $\mathbf{M}$. Let $H'=\int_0^\infty\lambda dE_\lambda$, then we have $H'E_n=H'_n\in\mathbf{M}$ since H' belongs to $\mathbf{M}$, and for every $A\in\mathbf{M}$: $\tau(A)=\lim_{n\to\infty}\varphi(AH'_n)$.

Using the above representation of φ, we have for $n=1,2,\ldots$

$$\tau(I)\geqq\varphi(H'_n)=\sum_{\alpha=1}^{\infty}\|H_n'^{\frac{1}{2}}x_\alpha\|^2.$$

This shows that every x_α $(\alpha=1,2,\ldots)$ is in the domain of definition of the (in general unbounded) operator $H'^{\frac{1}{2}}$, and putting $y_\alpha=H'^{\frac{1}{2}}x_\alpha$ $(\alpha=1,2,\ldots)$

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[6]) The countability of this system follows from the σ-finiteness of $\mathbf{M}$.

we have $\sum_{\alpha=1}^{\infty}\|y_\alpha\|^2<+\infty$. From this follows:

$$\tau(A)=\lim_{n\to\infty}\varphi(H_n'A)=\lim_{n\to\infty}\varphi(H_n'^{\frac{1}{2}}AH_n'^{\frac{1}{2}})=$$

$$=\lim_{n\to\infty}\sum_{\alpha=1}^{\infty}(AH_n'^{\frac{1}{2}}x_\alpha, H_n'^{\frac{1}{2}}x_\alpha)=\sum_{\alpha=1}^{\infty}(Ay_\alpha, y_\alpha)\quad (A\in\mathbf{M}).$$

Similarly, x_α $(\alpha=1,2,\ldots)$ lie in the domain of definition of the operator $H''^{\frac{1}{2}}$, and we have putting $z_\alpha=H''^{\frac{1}{2}}x_\alpha$ $(\alpha=1,2,\ldots)$

$$\varrho(A)=\sum_{\alpha=1}^{\infty}(Az_\alpha, z_\alpha)\quad (A\in\mathbf{M}).$$

To complete our proof it suffices to show the existence of a closed, densely defined operator T belonging to $\mathbf{M}$ such that $z_\alpha=Ty_\alpha$ $(\alpha=1,2,\ldots)$. To do this we proceed as follows. Since $\varrho\prec\tau$, $\tau(P)=0$ implies $\varrho(P)=0$ for $P\in\mathbf{M}_p$. But $\tau(P)=\lim_{n\to\infty}\varphi(H_n'P)$, and observing that $\varphi(P)=0$ implies $P=O$ for $P\in\mathbf{M}_p$ it follows $H_n'P=O$ $(n=1,2,\ldots)$ or $H'P=O$. Similarly $\varrho(P)=0$ for a $P\in\mathbf{M}_p$ implies $H''P=O$. Resuming, $\varrho\prec\tau$ implies $H''P=O$ $(P\in\mathbf{M}_p)$ if $H'P=O$, or $\overline{\text{Range } H'}\supseteq\overline{\text{Range } H''}$. But putting $J=\int_0^\infty\lambda^{-\frac{1}{2}}dE_\lambda$ $\left(H'=\int_0^\infty\lambda dE_\lambda\right)$ the closed extension T of $H''^{\frac{1}{2}}J$ plainly satisfies our requirements (for the existence of this extension see [3], p. 264, where the results of [4], pp. 221—229 are generalized to finite rings), for if P_0 is the projection on $\overline{\text{Range } H'}$ we have

$$Ty_\alpha=H''^{\frac{1}{2}}JH'^{\frac{1}{2}}x_\alpha=H''^{\frac{1}{2}}P_0x_\alpha=H''^{\frac{1}{2}}x_\alpha=z_\alpha\qquad(\alpha=1,2,\ldots),$$

qu. e. d.

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