On the contractive linear transformations of n-dimensional vector space.

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Notations.

a scalar	$\mathbf{a}^*, \mathbf{A}^*$ conjugate transposed of $\mathbf{a}, \mathbf{A}$
$\mathbf{a}$ column vector	$\mathbf{E}_k$ k-th order unit matrix
$\mathbf{A}=[a_{ij}]$ matrix	$\langle \mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m \rangle$ partitioned diagonal matrix

1. Consider a contractive linear transformation or contraction $\mathbf{T}$ of the n-dimensional vector space R_n, i. e. such that for every $\mathbf{x}$

$$\mathbf{x}^*\mathbf{T}^*\mathbf{T}\mathbf{x} \leqq \mathbf{x}^*\mathbf{x}.$$

It is known[1]) that a contraction can be represented as the product of a unitary transformation $\mathbf{U}$ in the $2n$-dimensional space R_{2n} and an orthogonal projection $\mathbf{P}$ on the original R_n. Using the symbols of matrix algebra, this assertion can be expressed as follows:

For any n-th order contractive matrix $\mathbf{T}$ there is a $2n$-th order unitary matrix $\mathbf{U}$ and a $2n$-th order orthogonal projector $\mathbf{P}$ such that

$$\mathbf{PU}=\begin{bmatrix}\mathbf{T} & * \\ * & *\end{bmatrix},$$

where the asterisks denote n-th order submatrices whose expressions are not needed. By the aid of these symbols the original transformation $\mathbf{x}^{(1)}=\mathbf{Tx}$ of R_n appears as a transformation of the subspace R_n of R_{2n}

$$\begin{bmatrix}\mathbf{x}^{(1)} \\ 0\end{bmatrix}=\mathbf{PU}\begin{bmatrix}\mathbf{x} \\ 0\end{bmatrix}=\begin{bmatrix}\mathbf{E}_n & 0 \\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{T} & * \\ * & *\end{bmatrix}\begin{bmatrix}\mathbf{x} \\ 0\end{bmatrix}=\begin{bmatrix}\mathbf{Tx} \\ 0\end{bmatrix}.$$

An analogous decomposition exists for the contractions of Hilbert space.

Recently B. SZ.-NAGY[2]) has found a similar representation simultaneously for all powers of a contraction and of its transposed. His theorem reads as follows:

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[1]) P. R. HALMOS, Normal dilations and extensions of operators, *Summa Brasiliensis Math.*, 2 (1950), 125—134.

[2]) B. SZ.-NAGY, Sur les contractions de l'espace de Hilbert, *these Acta*, 15 (1953), 87—92; Transformations de l'espace de Hilbert, fonctions de type positif sur un groupe, *these Acta*, 15 (1954), 104—114.

If $\mathbf{T}$ is a contraction of Hilbert space H, then there is a unitary transformation $\mathbf{U}$ of an other Hilbert space (which contains H as a subspace) such that for every vector $\begin{bmatrix}\mathbf{x}\\0\end{bmatrix}$ from H we have

$$\mathbf{T}^k\begin{bmatrix}\mathbf{x}\\0\end{bmatrix}=\mathbf{P}\mathbf{U}^k\begin{bmatrix}\mathbf{x}\\0\end{bmatrix},\ \mathbf{T}^{*k}\begin{bmatrix}\mathbf{x}\\0\end{bmatrix}=\mathbf{P}\mathbf{U}^{-k}\begin{bmatrix}\mathbf{x}\\0\end{bmatrix}\qquad (k=0,1,2,\ldots),$$

where $\mathbf{P}$ denotes the orthogonal projection on H.

In the present note I shall investigate with elementary tools the following finite analogon of B. Sz.-Nagy's problem: Is there a similar, but elementary matrix-representation for the first k positive powers of a contraction of R_n? The answer is given by the following

Theorem. *If $\mathbf{T}$ is a contraction of R_n, then R_n can be embedded in an $R_{(k+1)n}$ in such a way that we have*

$$\mathbf{P}\mathbf{U}^\varkappa=\underbrace{\begin{bmatrix}\mathbf{T}^\varkappa & *\\ * & *\end{bmatrix}}_{n\ \ nk}\begin{matrix}\}\,n\\ \}\,kn\end{matrix}\qquad (\varkappa=1,2,\ldots k)$$

where $\mathbf{U}$ is unitary, i.e. $\mathbf{U}\mathbf{U}^*=\mathbf{E}_{(k+1)n}$, and $\mathbf{P}$ is an orthogonal projector, i. e. $\mathbf{P}^*=\mathbf{P}$, $\mathbf{P}^2=\mathbf{P}$.

By the aid of these symbols the first, second, ..., k-th power of the original transformation $\mathbf{x}^{(1)}=\mathbf{T}\mathbf{x}$ of R_n appear as transformations of the subspace R_n of $R_{(k+1)n}$:

$$\begin{matrix}n\{\\ kn\{\end{matrix}\begin{bmatrix}\mathbf{x}^{(\varkappa)}\\0\end{bmatrix}=\mathbf{P}\mathbf{U}^\varkappa\begin{bmatrix}\mathbf{x}\\0\end{bmatrix}=\underbrace{\begin{bmatrix}\mathbf{E}_n & 0\\ 0 & 0\end{bmatrix}}_{n\ kn}\begin{bmatrix}\mathbf{T} & *\\ * & *\end{bmatrix}^\varkappa\begin{bmatrix}\mathbf{x}\\0\end{bmatrix}=\begin{bmatrix}\mathbf{T}^\varkappa\mathbf{x}\\0\end{bmatrix}\qquad (\varkappa=1,2,\ldots,k).$$

2. In order to prove our theorem, let us begin with an elementary study of the simplest case: $n=1$, $k=2$, $\mathbf{T}=\overline{\mathbf{T}}$. In this case $\mathbf{x}^{(1)}=\mathbf{T}\mathbf{x}$ means a contraction of R_1, i.e. of a straight line, which can be represented by

$$x^{(1)}=\cos\varphi\cdot x\qquad (\varphi \text{ real}),$$

and now we have to find a rotation $\mathbf{O}$ of R_3 such that

$$\left\{\begin{array}{l}\text{the orthogonal projection of } \mathbf{O}\begin{bmatrix}x\\0\\0\end{bmatrix} \text{ on } 0x \text{ is } \mathbf{T}\mathbf{x}=\cos\varphi\cdot x,\\[2ex] \quad ,,\qquad ,,\qquad ,,\qquad ,,\ \mathbf{O}^2\begin{bmatrix}x\\0\\0\end{bmatrix}\ ,,\ ,,\ ,,\ \mathbf{T}^2\mathbf{x}=\cos^2\varphi\cdot x.\end{array}\right.$$

The rotation $\mathbf{O}_z=\begin{bmatrix}\cos\varphi & \sin\varphi & 0\\ -\sin\varphi & \cos\varphi & 0\\ 0 & 0 & 1\end{bmatrix}$ brings the point $\begin{bmatrix}x\\0\\0\end{bmatrix}$ into $\begin{bmatrix}x\cos\varphi\\ x\sin\varphi\\ 0\end{bmatrix}$

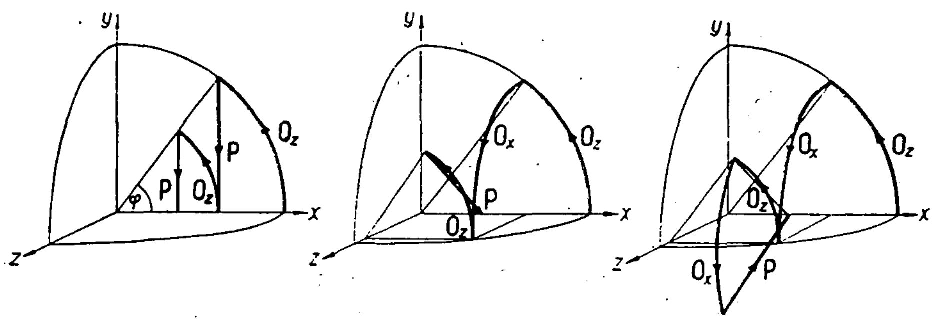
whose orthogonal projection on $0x$ is

$$\mathbf{PO}_z\begin{bmatrix}x\\0\\0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\cos\varphi\\x\sin\varphi\\0\end{bmatrix}=\begin{bmatrix}x\cos\varphi\\0\\0\end{bmatrix}=\begin{bmatrix}\mathbf{Tx}\\0\\0\end{bmatrix}.$$

Hence

$$\mathbf{PO}_z=\begin{bmatrix}\cos\varphi&*&*\\ *&*&*\\ *&*&*\end{bmatrix}.$$

Denote the rotation around the axis $0x$ through the angle $\frac{\pi}{2}$ by $\mathbf{O}_x$, and the orthogonal projection on $0x$ by $\mathbf{P}$. Further compare the three figures below:



An inspection of these figures shows clearly, that the transformation $\mathbf{T}^2$ i.e. $x^{(2)}=\cos^2\varphi\cdot x$ can be decomposed into rotations and projections in different manners:

$$\begin{bmatrix}\mathbf{T}^2\mathbf{x}\\0\\0\end{bmatrix}=\mathbf{PO}_z\mathbf{PO}_z\begin{bmatrix}x\\0\\0\end{bmatrix}=\mathbf{PO}_z\mathbf{O}_x\mathbf{O}_z\begin{bmatrix}x\\0\\0\end{bmatrix}=\mathbf{PO}_x\mathbf{O}_z\mathbf{O}_x\mathbf{O}_z\begin{bmatrix}x\\0\\0\end{bmatrix}.$$

The first one is a simple iteration of the previously found decomposition $\mathbf{T}=\mathbf{PO}_z$, while the third furnishes the representation $\mathbf{T}^2=\mathbf{PO}^2$ with $\mathbf{O}=\mathbf{O}_x\mathbf{O}_z$. We have at the same time $\mathbf{T}=\mathbf{PO}=\mathbf{PO}_x\mathbf{O}_z$, because obviously $\mathbf{PO}_x=\mathbf{P}$.

Consequently the rotation $\mathbf{O}$ which satisfies the conditions (1) is given by

$$\mathbf{O}=\mathbf{O}_x\mathbf{O}_z=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}\cos\varphi&\sin\varphi&0\\-\sin\varphi&\cos\varphi&0\\0&0&1\end{bmatrix}=\begin{bmatrix}\cos\varphi&\sin\varphi&0\\0&0&-1\\-\sin\varphi&\cos\varphi&0\end{bmatrix}.$$

Thus we proved that the projection of the first and second power of the rotation $\mathbf{O}$, applied on the points (vectors) of the $0x$ axis furnish the first and second power of the contraction $\mathbf{T}$.

This intuitive treatment suggests that in order to find a similar representation for the three first powers of $\mathbf{T}$ we need the four dimensional space, and a convenient orthogonal matrix will be found by multiplying $\begin{bmatrix} \mathbf{O}_x\mathbf{O}_z & 0 \\ 0 & 1 \end{bmatrix}$ from the left by $\mathbf{O}_w$, i. e. by the matrix of the rotation round the fourth axis $0w$ through the angle $\frac{\pi}{2}$. We get in this way

$$\mathbf{O} = \mathbf{O}_w \langle \mathbf{O}_x \mathbf{O}_z, 1 \rangle = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\varphi & \sin\varphi & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -\sin\varphi & \cos\varphi & 0 & 0 \end{bmatrix}.$$

$\mathbf{O}$ is obviously orthogonal, being the product of orthogonal matrices, and an easy calculation shows that $\mathbf{O}$ satisfies all the requirements of the problem.

We generalise now this result to the case $n=1$, k arbitrary, and prove that

$$\text{(2)} \qquad k\left\{\begin{bmatrix} \mathbf{T}^\varkappa x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right. = \begin{bmatrix} \cos^\varkappa \varphi \cdot x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{P}\mathbf{O}^\varkappa \begin{bmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (\varkappa = 1, 2, \ldots, k)$$

where $\mathbf{P} = \langle 1, \underbrace{0, \ldots 0}_{k} \rangle$, and

$$\mathbf{O} = \left.\begin{bmatrix} c & s & 0 & 0 & \ldots & 0 \\ 0 & 0 & -1 & 0 & \ldots & 0 \\ 0 & 0 & 0 & -1 & \ldots & 0 \\ & & & & & \\ 0 & 0 & 0 & 0 & \ldots & -1 \\ -s & c & 0 & 0 & \ldots & 0 \end{bmatrix}\right\} k+1, \quad c = \cos\varphi, s = \sin\varphi.$$

Obviously, $\mathbf{O}\mathbf{O}^* = \mathbf{E}_{k+1}$, thus $\mathbf{O}$ is a rotation of R_{k+1}, and we have only to show that the element in the upper left corner of $\mathbf{O}^\varkappa$ is equal to $c^\varkappa = \cos^\varkappa \varphi$, when $\varkappa \leqq k$. Denote the elements of $\mathbf{O}^\varkappa$ by $a_{ij}^{(\varkappa)}$ $(i, j = 1, 2, \ldots, k+1)$. Since

$$a_{11}^{(1)} = c, \quad a_{12}^{(1)} = s, \quad a_{k+1,1}^{(1)} = -s, \quad a_{k+1,2}^{(1)} = c, \quad a_{23}^{(1)} = a_{34}^{(1)} = \cdots = a_{k,k+1}^{(1)} = -1,$$

$$\text{each other } a_{ij}^{(1)} = 0,$$

the application of the relations

$$a_{ij}^{(\varkappa)} = a_{i1}^{(\varkappa-1)} a_{1j}^{(1)} + a_{i2}^{(\varkappa-1)} a_{2j}^{(1)} + \cdots + a_{i,k+1}^{(\varkappa-1)} a_{k+1,j}^{(1)}$$

proves, by induction,

$$a_{1,k+1}^{(\varkappa)} = -a_{1,k}^{(\varkappa-1)} = \cdots = \pm a_{1,k-\varkappa+2}^{(1)} = 0 \quad \text{for} \quad \varkappa \leqq k-1,$$

and

$$a_{11}^{(\varkappa+1)}=a_{11}^{(\varkappa)}a_{11}^{(1)}+a_{1,k+1}^{(\varkappa)}a_{k+1,1}^{(1)}=a_{11}^{(\varkappa)}a_{11}^{(1)} \quad \text{for} \quad \varkappa\leqq k-1.$$

Hence

$$a_{11}^{(\varkappa)}=a_{11}^{\varkappa}=\cos^{\varkappa}\varphi \quad \text{for} \quad \varkappa\leqq k. \tag{2.1}$$

Q. e. d.

Thus we proved that the projections of the first, second, ..., k-th power of the rotation $\mathbf{O}$, applied on the points of the $0x_1$ axis, furnish the first, second, ..., k-th power of the contraction $x_1^{(1)}=\cos\varphi\cdot x_1$.

3. Let us now consider the general case of a contractive transformation of R_n. Let $\mathbf{T}=\mathbf{C}\mathbf{V}$ be the polar factorisation, where $\mathbf{C}=(\mathbf{T}\mathbf{T}^*)^{1/2}$ is (semi-) definite and $\mathbf{V}$ is unitary.

$\mathbf{C}$ as well as $\mathbf{T}$ being contractive, there is a uniquely determined positive (semi-) definite square-root of $\mathbf{E}_n-\mathbf{C}^2$, which we denote by $\sqrt{\mathbf{E}_n-\mathbf{C}^2}$. Consider now the $(k+1)n$-th order matrix $\mathbf{U}$ in the partitioned form

$$\mathbf{U}=\begin{bmatrix} \mathbf{C}\mathbf{V} & \sqrt{\mathbf{E}_n-\mathbf{C}^2}\,\mathbf{V} & 0 & 0 & \dots & 0 \\ 0 & 0 & -\mathbf{E}_n & 0 & \dots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \dots & -\mathbf{E}_n \\ -\sqrt{\mathbf{E}_n-\mathbf{C}^2}\,\mathbf{V} & \mathbf{C}\mathbf{V} & 0 & 0 & \dots & 0 \end{bmatrix}, \quad \mathbf{U}\mathbf{U}^*=\mathbf{E}_{(k+1)n}. \tag{3}$$

We write the vectors of $R_{(k+1)n}$ in the partitioned form $\begin{bmatrix}\mathbf{x}_1\\ \vdots \\ \mathbf{x}_{k+1}\end{bmatrix}$ where $\mathbf{x}_1$ is the component belonging to the subspace R_n.

With these notations we can write at once the analogon of the equations (2)

$$\begin{bmatrix}\mathbf{T}^{\varkappa}\mathbf{x}_1\\ 0\\ \vdots\\ 0\end{bmatrix}=\mathbf{P}\mathbf{U}^{\varkappa}\begin{bmatrix}\mathbf{x}_1\\ 0\\ \vdots\\ 0\end{bmatrix} \qquad (\varkappa=1,2,\dots,k), \tag{4}$$

where $\mathbf{P}=\langle\mathbf{E}_n,0,\dots,0\rangle$ and the unitary matrix $\mathbf{U}$ is given by (3). Due to the coincidence of the multiplication-rules for ordinary matrices and hypermatrices, the proof given for the equations (2) remains valid for the equations (4) if one replaces the elements $c, s, 1, 0$, by the submatrices $\mathbf{C}\mathbf{V}$, $\sqrt{\mathbf{E}_n-\mathbf{C}^2}\,\mathbf{V}$, $\mathbf{E}_n$, 0, respectively. So the proof of our Theorem is complete.

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