On the Prüfer manifold and a problem of Alexandroff and Hopf.

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1. Let L denote the real line, taken with discrete topology, and let P be the upper half-plane $y>0$ of the (x, y)-plane, with its usual topology. Let M be the topological product $L \times P$, and let W denote the nontriangulable manifold of PRÜFER [2; 3, p. 71].

Theorem. *M is a nonseparable metrizable space. W is a connected, nonnormal, regular, locally Euclidean space. W is a decomposition space of M, and the corresponding identification map $\varphi: M \to W$ is a continuous, open, local homeomorphism.*

Remarks. i) The nonnormality of W is a stronger result than the known fact [2] that the 2nd countability axiom fails to hold on W. For, had W a countable base of open sets, so would its regularity imply, by a well known theorem of TYCHONOFF [4], that W be normal.

ii) P. ALEXANDROFF and H. HOPF have raised the question [1, p. 70] whether there exists or not a regular, nonnormal decomposition space of a normal space. Since the space M is obviously normal, their problem is answered in the affirmative. Moreover, the space M is not only normal, but even metrizable, while the decomposition space W is not only regular, but even completely regular. The identification map $\varphi: M \to W$ is not only strongly continuous (stark stetig [1, p. 65]); it is even an open local homeomorphism.

iii) It is known that, if X is normal and $f: X \to Y$ is continuous and closed, then Y is also normal. Our result shows that this need not hold any longer when closed is replaced by open.

2. It is obvious that $M = L \times P$ is a nonseparable metrizable space.

We shall use the complex number notation for points in P, i. e. $z = x + iy$, with $i = \sqrt{-1}$.

For each $t \in L$, let $f_t: P \to P$ be defined by

$$f_t(z) = z + \frac{z-t}{|z-t|}.$$

This is a homeomorphism mapping P on the open set $P(t)$ consisting of the points $z \in P$ which satisfy $|z-t|>1$. By means of the formula

$$f_t^{-1}(A)=f_t^{-1}[A \cap P(t)],$$

valid for any $A \subset P$, we shall frequently simplify the notations in the sequel.

For any $(t_0, z_0) \in M$, define

$$C(t_0, z_0)=\{(t_0, z_0)\} \cup \{(t, f_t f_{t_0}^{-1}(z_0)) \mid t \in L\}.$$

If $|z_0-t_0| \leqq 1$, then $f_{t_0}^{-1}(z_0)$ is empty, hence

$$C(t_0, z_0)=\{(t_0, z_0)\};$$

if $|z_0-t_0|>1$, then any (t, z) in $C(t_0, z_0)$ satisfies $|z-t|>1$ and

$$C(t_0, z_0)=\{(t, z) \mid f_t^{-1}(z)=f_{t_0}^{-1}(z_0)\}.$$

Any $(t_0, z_0) \in M$ belongs to $C(t_0, z_0)$ and it is readily seen that any two $C(t_j, z_j)$, $j=1, 2$, are either disjoint or identical. Thus we have a decomposition of the space M into disjoint closed subsets. We define[1]) the PRÜFER manifold W as the associated decomposition space and denote the corresponding identification map by

$$\varphi: M \rightarrow W.$$

Any $C(t_0, z_0)$ meets each $t \times P \subset M$ in at most one point. For any $A \subset P$ we have

$$(1) \qquad (t \times P) \cap \varphi^{-1}\varphi(t_0 \times A)=\begin{cases} t_0 \times A & \text{if } t=t_0 \\ t \times f_t f_{t_0}^{-1}(A) & \text{if } t \neq t_0, \end{cases}$$

hence

$$(2) \qquad \varphi^{-1}\varphi(t_0 \times A)=(t_0 \times A) \cup \bigcup_{t \in L}(t \times f_t f_{t_0}^{-1}(A)).$$

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[1]) According to T. RADÓ [2, p. 107—110], the PRÜFER manifold R consists of real points, having three real coordinates (x, y, a) subject to the conditions $y>0$, $(x-a)^2+y^2 \leq 1$, and imaginary points (x, y, i) where x, y are reals with $y>0$, while $i=\sqrt{-1}$. For each real a, let $B(a)$ be the set of all the points of R, having a or i as their third coordinate; let also T_a be the one-one transformation of $B(a)$ onto the upper half-plane P, defined by $T_a(x, y, a)=z$ and $T_a(x, y, i)=f_a(z)$, where $z=x+iy$ and f_a is the same as above. A topology is introduced in R, according to which each $B(a)$ is open in R, while T_a is a homeomorphism of $B(a)$ with P [2, p. 110].

It is now a simple matter to realize the equivalence between RADÓ's definition and the one given in the present paper.

Let S be the transformation of $M=L \times P$ onto R, defined by

$$S(t, z)=\begin{cases} (x, y, t) & \text{with } z=x+iy, \text{ if } |z-t| \leq 1, \\ (x', y', i) & \text{with } f_t^{-1}(z)=x'+iy', \text{ if } |z-t|>1. \end{cases}$$

The transformation S_a of P onto $B(a)$, defined by $S_a(z)=S(a, z)$, is one-one; its inverse is RADÓ's T_a, i. e. $S_a^{-1}=T_a$. Since T_a is a homeomorphism, so is S_a; since each $B(a)$ is open in R, it follows that S is continuous and open. Setting now $\psi=S\varphi^{-1}$, we obtain the desired homeomorphism of W onto R.

3. Since W is obtained from M by topological identification, φ is a continuous map. It is also an open map; for, if $U \subset P$ is open, since each f_t is continuous and open, (2) implies that the set $\varphi^{-1}\varphi(t \times U)$ is open in M, hence $\varphi(t \times U)$ is open in W.

Since each $C(t_0, z_0)$ meets $t \times P$ in at most one point, φ maps each open subset $t \times P$ of M topologically onto the open subset $\varphi(t \times P)$ of W. It follows that φ is a local homeomorphism.

4. Since each $p \in W$ belongs to some $\varphi(t \times P)$, which is open in W and homeomorphic to a plane, the space W is locally Euclidean. Since, for example, the point $\varphi(0, 2i)$ belongs to each $\varphi(t \times P)$, which is a connected subset, the space W is connected.

5. W is a Hausdorff space. For, let

$$p_1 = \varphi(t_1, z_1) \neq \varphi(t_2, z_2) = p_2.$$

If $t_1 = t_2 = t$, then $z_1 \neq z_2$; let $U_j \ni z_j$ $(j = 1, 2)$ be open in P, satisfying $U_1 \cap U_2 = \emptyset$, where $\emptyset$ is the empty set. It follows that $p_j \in \varphi(t \times U_j)$, which is open in W $(j = 1, 2)$ and, since φ is univalent on $t \times P$,

$$\varphi(t \times U_1) \cap \varphi(t \times U_2) = \emptyset.$$

Let us now assume that $t_1 \neq t_2$.

If, for example, $|z_1 - t_1| > 1$ holds, then let

$$z_2' = f_{t_2} f_{t_1}^{-1}(z_1),$$

hence

$$\varphi(t_2, z_2') = \varphi(t_1, z_1) = p_1 \neq p_2 = \varphi(t_2, z_2)$$

and the distinct points p_1, p_2 may be separated by open sets as above.

If, on the contrary, both $|z_1 - t_1| \leqq 1$ and $|z_2 - t_2| \leqq 1$ hold, then $t_1 \neq t_2$ implies, for example, $t_2 - t_1 = 2r > 0$ and the open subset $U_j = \{z \mid |z - t_j| < 1 + r\}$ of P contains z_j $(j = 1, 2)$. As a consequence, p_j belongs to the open subset $\varphi(t_j \times U_j)$ of W and

$$\varphi(t_1 \times U_1) \cap \varphi(t_2 \times U_2) = \emptyset$$

holds. For, the presence of a $\zeta_j \in U_j$ $(j = 1, 2)$ satisfying

$$\varphi(t_1, \zeta_1) = \varphi(t_2, \zeta_2)$$

implies, since $t_1 \neq t_2$,

$$|\zeta_j - t_j| > 1 \quad \text{and} \quad f_{t_1}^{-1}(\zeta_1) = f_{t_2}^{-1}(\zeta_2).$$

It follows that

$$\zeta_1 - \frac{\zeta_1 - t_1}{|\zeta_1 - t_1|} = \zeta_2 - \frac{\zeta_2 - t_2}{|\zeta_2 - t_2|} = \alpha,$$

$$|\alpha - t_j| = |\zeta_j - t_j| - 1 < r,$$

hence

$$|t_2 - t_1| \leqq |t_2 - \alpha| + |\alpha - t_1| < 2r,$$

which contradicts the definition of r.

6. Since W is a Hausdorff locally Euclidean space, it is locally compact, regular and completely regular.

7. It remains to show that W is not normal. For this purpose, let L_1, L_2 be the sets of rational, respectively irrational numbers of L. The set

$$A_j = \{(t, t+i) | t \in L_j\} \text{ is closed in } M \text{ and } A_1 \cap A_2 = \emptyset.$$

Let $F_j = \varphi(A_j)$, $j = 1, 2$; from $|t+i-t| = 1$ it follows that $\varphi^{-1}(F_j) = A_j$, thus, for $j = 1, 2$,

$$F_j \text{ is closed in } W \text{ and } F_1 \cap F_2 = \emptyset.$$

Let now V_j be any open subset of W containing F_j ($j = 1, 2$); we shall prove that

(3) $$V_1 \cap V_2 \neq \emptyset.$$

In fact, for each $t \in L$ there exists an open disk $D_t \subset P$, centered in $t+i$ and such that

$$t \times D_t \subset \varphi^{-1}(V_j),$$

hence

$$\varphi^{-1}\varphi(t \times D_t) \subset \varphi^{-1}(V_j) \quad \text{if} \quad t \in L_j.$$

For any $t \in L$, let

$$H_t = f_t^{-1}(D_t) \subset P \quad \text{and} \quad G_j = \bigcup_{t \in L_j} H_t.$$

Formula (1) implies that

$$0 \times f_0(G_j) \subset \varphi^{-1}\varphi(\bigcup_{t \in L_j} t \times D_t),$$

hence

$$0 \times f_0(G_1 \cap G_2) \subset \varphi^{-1}(V_1) \cap \varphi^{-1}(V_2).$$

Finally, (3) is a consequence of

$$\varphi^{-1}(V_1) \cap \varphi^{-1}(V_2) \neq \emptyset,$$

which follows from the inequality

$$G_1 \cap G_2 \neq \emptyset,$$

which we proceed now to prove.

For $j = 1, 2$ and $n, l = 1, 2, \ldots$ let

$$M_j^n = \left\{ t \,\middle|\, t + \frac{i}{n} \in G_j \right\} \quad \text{and} \quad N_j^l = \bigcup_{n \geqq l} M_j^n.$$

It is obvious that, for each $t \in L_j$, $t + \frac{i}{n} \in G_j$, hence $t \in M_j^n$ holds for almost all the indices n; as a consequence

(4) $$L_j \subset \cap N_j^l.$$

Furthermore, $t \in \cap N_j^l$ implies $t + \frac{i}{n_l} \in G_j$, where $n_l \geqq l$ for each $l \geqq 1$. Assuming now

(5) $$G_1 \cap G_2 = \emptyset,$$

we obtain $t \in L_j$, hence

(6) $$\cap N_j^l \subset L_j,$$

since $t \in L_{3-j}$ implies $t + \frac{i}{n} \in G_{3-j}$ for almost all the indices n.

Since G_j is open in P, M_j^n and N_j^l are open subsets of the real line R in its usual topology. On account of (4) and (6), we see that (5) implies that L_1, the set of rational numbers, is a G_δ in R, which is known to be impossible: the nonnormality of the space W is thereby completely proved.

(Added in proof:) The author realized recently that the nonnormality of the Prüfer manifold has already been proved by E. Calabi and M. Rosenlicht in their paper: Complex analytic manifolds without countable base, *Proceedings American Math. Soc.*, **4** (1953), 335—340.

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