On a problem in set theory.

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P. ERDŐS has posed the following problem:

Let $\mathfrak{m}$ be an infinite cardinal number, φ the initial number of power $\mathfrak{m}$, and $\Gamma_{\mathfrak{m}}$ the set of ordinal numbers less than φ. Further let $\mathfrak{n}$ be a given cardinal number which is smaller than $\mathfrak{m}$. Suppose S is a given set of power $\mathfrak{m}$, and that to every element γ of $\Gamma_{\mathfrak{m}}$ there corresponds a subset $S(\gamma)$ of S such that $\overline{\overline{S(\gamma)}} < \mathfrak{n}$. Problem: Is there a subset Γ of power $\mathfrak{m}$ of $\Gamma_{\mathfrak{m}}$ such that

$$\overline{\overline{S - \bigcup_{\gamma \in \Gamma} S(\gamma)}} = \mathfrak{m} \ ?$$

If we replace the condition $\mathfrak{n} < \mathfrak{m}$ by $\mathfrak{n} \leqq \mathfrak{m}$, the answer is in general negative. Indeed, let $S = \Gamma_{\mathfrak{m}}$ and define the set $S(\gamma)$ as the set of all $\beta < \gamma$. Clearly for any subset Γ of power $\mathfrak{m}$ of $\Gamma_{\mathfrak{m}}$ we have

$$S - \bigcup_{\gamma \in \Gamma} S(\gamma) = 0.$$

ERDŐS has proved[1]) that the answer to the problem (with $\mathfrak{n} < \mathfrak{m}$) is in the affirmative, but his proof uses the generalized continuum hypothesis.

We shall give in this paper a proof without using the generalized continuum hypothesis.[2])

First we prove the following

Lemma 1. *Let $\mathfrak{q}$ be a regular cardinal number and $\mathfrak{p}$ a cardinal number which is smaller than $\mathfrak{q}$. If to every element γ of $\Gamma_{\mathfrak{q}}$ there corresponds an ordinal number $g(\gamma) \in \Gamma_{\mathfrak{p}}$, then there exists an ordinal number $\pi \in \Gamma_{\mathfrak{p}}$ and a subset Γ of power $\mathfrak{q}$ of $\Gamma_{\mathfrak{q}}$ such that for every element γ of Γ we have $g(\gamma) < \pi$.*

Proof. Let $H(\alpha)$ denote for every ordinal number $\alpha \in \Gamma_{\mathfrak{p}}$ the set of all $\gamma \in \Gamma_{\mathfrak{q}}$ for which $g(\gamma) = \alpha$. It is clear that

$$\Gamma_{\mathfrak{q}} = \bigcup_{\alpha \in \Gamma_{\mathfrak{p}}} H(\alpha).$$

As $\mathfrak{p} < \mathfrak{q}$ and $\mathfrak{q}$ is regular it follows that there exists an ordinal number $\pi' \in \Gamma_{\mathfrak{p}}$ for which $\overline{\overline{H(\pi')}} = \mathfrak{q}$. By the definition of $H(\alpha)$ the lemma holds with $\Gamma = H(\pi')$ and $\pi = \pi' + 1$.

[1]) P. ERDŐS, Some remarks on set theory. III, *Michigan Math. Journal*, **2** (1953/54), 51—57.

[2]) I am indepted to P. ERDŐS and L. GILLMAN who after reading the first draft of this paper simplified my original proof.

Lemma 2. *Let* $\mathfrak{a}$ *be a transfinite singular cardinal number and* $\mathfrak{b}$ *a regular cardinal number which is smaller than* $\mathfrak{a}$. *Further let* $\mathfrak{r}$ *denote the smallest cardinal number such that* $\mathfrak{a}$ *is the sum of* $\mathfrak{r}$ *cardinal numbers each of which is less than* $\mathfrak{a}$. *If* $\mathfrak{b} > \mathfrak{r}$ *and to every element* γ *of* $\Gamma_{\mathfrak{a}}$ *there corresponds an ordinal number* $h(\gamma) \in \Gamma_{\mathfrak{b}}$ *then there exists an ordinal number* $\pi \in \Gamma_{\mathfrak{b}}$ *and a subset* Γ *of power* $\mathfrak{a}$ *of* $\Gamma_{\mathfrak{a}}$ *such that for every element* γ *of* Γ *we have* $h(\gamma) < \pi$.

Proof. Let μ denote the initial number of $\mathfrak{r}$. There exist regular cardinal numbers $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_\xi, \ldots$ $(\xi < \mu)$ such that $\mathfrak{a}_\beta > \mathfrak{a}_\alpha$ for $\beta > \alpha$, $\mathfrak{b} < \mathfrak{a}_\xi < \mathfrak{a}$ for every $\xi < \mu$ and

$$\mathfrak{a} = \sum_{\xi < \mu} \mathfrak{a}_\xi .$$

By lemma 1 there exists for every ordinal number $\xi < \mu$ an ordinal number $\pi_\xi \in \Gamma_{\mathfrak{b}}$ such that there are $\mathfrak{a}_\xi$ ordinal numbers $\gamma \in \Gamma_{\mathfrak{a}}$ satisfying $h(\gamma) < \pi_\xi$. Indeed, let $\mathfrak{q} = \mathfrak{a}_\xi$, $\mathfrak{p} = \mathfrak{b}$ and $g(\gamma) = h(\gamma)$ on $\Gamma_{\mathfrak{a}_\xi}$. As $\mathfrak{b} < \mathfrak{a}_\xi$ and $\mathfrak{a}_\xi$ is regular, the conditions of lemma 1 are fulfilled. Accordingly, there exists an ordinal number $\pi_\xi \in \Gamma_{\mathfrak{b}}$ and a subset Γ_ξ of power $\mathfrak{a}_\xi$ of $\Gamma_{\mathfrak{a}_\xi}$ such that for every element γ of Γ_ξ we have $g(\gamma) = h(\gamma) < \pi_\xi$.

Since $\mathfrak{r} < \mathfrak{b}$ and $\mathfrak{b}$ is regular there exists an ordinal number $\pi \in \Gamma_{\mathfrak{b}}$ for which $\pi_\xi < \pi$ for every ξ, $0 \leqq \xi < \mu$. Let

$$\Gamma = \bigcup_{\xi < \mu} \Gamma_\xi .$$

Clearly the power of Γ is $\mathfrak{a}$. Let $\gamma \in \Gamma$. It follows that $\gamma \in \Gamma_{\xi_0}$ for some ordinal number $\xi_0 < \mu$. By the definition of Γ_ξ we have $h(\gamma) < \pi_{\xi_0}$. As $\pi_\xi < \pi$ and $\pi \in \Gamma_{\mathfrak{b}}$ the lemma 2 is proved.

Theorem. *Let* S *be a given infinite set of power* $\mathfrak{m}$, *and* $\mathfrak{n}$ *a given cardinal number which is smaller then* $\mathfrak{m}$. *If to every element* γ *of* $\Gamma_{\mathfrak{m}}$ *there corresponds a subset* $S(\gamma)$ *of* S *such that* $\overline{\overline{S(\gamma)}} < \mathfrak{n}$, *then there exists a subset* Γ *of power* $\mathfrak{m}$ *of* $\Gamma_{\mathfrak{m}}$ *for which*

$$\overline{\overline{S - \bigcup_{\gamma \in \Gamma} S(\gamma)}} = \mathfrak{m}.$$

Proof. If there exists a regular cardinal number $\mathfrak{s}$ for which $\mathfrak{m} > \mathfrak{s} \geqq \mathfrak{n}$, decompose S into the union of $\mathfrak{s}$ disjoint sets $M_\varkappa$, of power $\mathfrak{m}$,

$$S = \bigcup_{\varkappa \in \Gamma_{\mathfrak{s}}} M_\varkappa .$$

Since $\overline{\overline{S(\gamma)}} < \mathfrak{n}$ and $\mathfrak{s}$ $(\geqq \mathfrak{n})$ is regular, there exists an $f(\gamma) \in \Gamma_{\mathfrak{s}}$ so that for any $\varkappa > f(\gamma)$, $M_\varkappa \cap S_\gamma$ is empty. Thus, for a suitable $\mathfrak{s}$, there exist by our lemmas an ordinal number $\pi \in \Gamma_{\mathfrak{s}}$ and a subset Γ of power $\mathfrak{m}$ of $\Gamma_{\mathfrak{m}}$ such that for every element γ of Γ we have $f(\gamma) < \pi$, hence

$$S - \bigcup_{\gamma \in \Gamma} S(\gamma) \supset \bigcup_{\pi \leqq \varkappa \in \Gamma_{\mathfrak{s}}} M_\varkappa$$

i. e.

$$\overline{\overline{S - \bigcup_{\gamma \in \Gamma} S(\gamma)}} = \mathfrak{m}.$$

Suppose now that there is no regular cardinal number $\mathfrak{s}$, for which $\mathfrak{n} \leqq \mathfrak{s} < \mathfrak{m}$. In this case $\mathfrak{m}$ is obviously regular.

Denote by N the set of all elements x of S for which $x \in S(\gamma)$ for every ordinal number γ, $\gamma \in \Gamma_{\mathfrak{m}}$. As $\overline{\overline{S(\gamma)}} < \mathfrak{n}$ $(\gamma \in \Gamma_{\mathfrak{m}})$, we have $\overline{\overline{N}} < \mathfrak{n}$. Let

$$(1) \qquad x_0, x_1, x_2, \ldots, x_\omega, x_{\omega+1}, \ldots, x_\zeta, \ldots (\zeta < \varphi)$$

be any well-ordering of $X = S - N$ of the type φ. We shall define by transfinite induction a (single-valued) mapping $H(x)$ of the set X on the set $\{S(\gamma)\}_{\gamma \in \Gamma_{\mathfrak{m}}}$ in the following manner: Let γ_0 be the smallest ordinal number $\gamma \in \Gamma_{\mathfrak{m}}$ for which $x_0 \notin S(\gamma)$, the existence of such a γ follows from the fact that $x_0 \in X$. Put $H(x_0) = S(\gamma_0)$. Let β be an arbitrary ordinal number, $1 \leqq \beta < \varphi$, and suppose that γ_ζ and $H(x_\zeta)$ are defined for every $\zeta < \beta$. If there is an ordinal number $\gamma \neq \gamma_\zeta$ $(\zeta < \beta)$, $\gamma \in \Gamma_{\mathfrak{m}}$, for which $x_\beta \notin S(\gamma)$, then let γ_β be the the smallest such ordinal number and let $H(x_\beta) = S(\gamma_\beta)$. In the opposite case, i. e. if $x_\beta \in S(\gamma)$ for any $\gamma \neq \gamma_\zeta$ $(\zeta < \beta)$, $\gamma \in \Gamma_{\mathfrak{m}}$, then let ζ_0 be the smallest ordinal number ζ $(\zeta < \beta)$ for which $x_\beta \notin S(\gamma_\zeta)$ and let $\gamma_\beta = \gamma_{\zeta_0}$, $H(x_\beta) = S(\gamma_{\zeta_0})$. The existence of such a ζ follows from the fact that $x_\beta \in X$.

Let $\alpha \in \Gamma_{\mathfrak{m}}$. We prove that

(i) if $A_\alpha = \{\eta_\zeta\}$ is the set of those $\eta \in \Gamma_{\mathfrak{m}}$ for which $\alpha = \gamma_\eta$, arranged in their natural order, then the power of A_α is smaller than $\mathfrak{n}$.

Suppose the contrary i. e. $\overline{\overline{A_\alpha}} \geqq \mathfrak{n}$. Let ψ be the initial number of $\mathfrak{n}$ and ϱ_0 the smallest ordinal number ϱ for which $\eta_\zeta < \varrho$ for every $\zeta < \psi$. Obviously $\varrho_0 \in \Gamma_{\mathfrak{m}}$, because $\mathfrak{m}$ is regular and $\overline{\overline{\psi}} = \mathfrak{n} < \mathfrak{m}$. Let μ be an element of $\Gamma_{\mathfrak{m}}$ for which $\mu \neq \gamma_\xi$ $(\xi < \varrho_0)$. By the definition of the set A_α and the mapping $H(x)$ we have $x_{\eta_\zeta} \in S(\mu)$ if $1 \leqq \zeta < \psi$. This is impossible, since $\overline{\overline{S(\gamma)}} < \mathfrak{n}$ $(\gamma \in \Gamma_{\mathfrak{m}})$.

By a theorem of S. PICCARD[3]), there exists a subset R of power $\mathfrak{m}$ of X for which

$$(2) \qquad R \cap H(R)^{4)} = 0.$$

The set R is well-ordered according to (1). Let $R = \{x_{\beta_\xi}\}_{\xi < \varphi}$. By (i) the power of the set Γ of all distinct γ_{β_ξ}'s $(\xi < \varphi)$ is $\mathfrak{m}$. According to (2)

$$R \subseteqq S - H(R) = S - \bigcup_{\xi < \varphi} S(\gamma_{\beta_\xi}) = S - \bigcup_{\gamma \in \Gamma} S(\gamma).$$

As $\overline{\overline{R}} = \overline{\overline{S}}$ we obtain that

$$\overline{\overline{S - \bigcup_{\gamma \in \Gamma} S(\gamma)}} = \mathfrak{m}.$$

The theorem is proved.

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[3]) We mean the following theorem. Let $\mathfrak{p}$ be a regular cardinal number, $\mathfrak{p} \geqq \aleph_0$, and E a set of power $\mathfrak{p}$. If to every element $x \in E$ there corresponds a subset $E(x)\,(x \notin E(x))$ of E such that for any $x \in E$ the power of the set $E(x)$ is smaller than a given cardinal number $\mathfrak{q}$ which is smaller than $\mathfrak{p}$, then E has a subset E' of power $\mathfrak{p}$ for which $E' \cap E(E') = 0$. [SOPHIE PICCARD, Sur un problème de M. Ruziewicz de la théorie des relations, *Fundamenta Math.*, 29 (1937), 5—9.]

[4]) $H(R) = \bigcup_{x \in R} H(x)$.