On the number of isomorphic classes of nonnormal subgroups in a finite group.

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Let G be a finite group. Let $r(G)$ be the number of isomorphic classes of nonnormal subgroups in G and let $t(G)$ be the number of distinct prime factors of the order of G. Recently TROFIMOV [2][1]) obtained the following results: (1) If $r(G) < t(G)+2$, then G is soluble. (2) If $r(G) < 7$, then G is soluble. He remarked also the following: If A_5 is the alternating group of degree 5, then $r(A_5) = 2t(A_5)+1 = 7$.

In this note we shall prove the following theorem, which contains the results of TROFIMOV as a special case.

Theorem. *If G is insoluble and if $r(G) < 2t(G)+2$, then G is isomorphic to A_5.*

Proof. We prove this theorem by an induction argument with respect to the order of the group.

(i) G contains no normal p-subgroup P which is distinct from a p-Sylow subgroup of G. In fact, otherwise, put $\bar{G} = G/P$. Then $\bar{G}$ is insoluble and $t(\bar{G}) = t(G)$. If $\bar{G}$ is not isomorphic to A_5, then $\bar{G}$ contains at least $2t(\bar{G})+2 = 2t(G)+2$ classes of nonnormal subgroups, and a fortiori G does so. Thus $\bar{G}$ is isomorphic to A_5. Then, as TROFIMOV remarked, $\bar{G}$ contains seven classes of nonnormal subgroups. Let q be a prime factor of the order of G distinct from p and let Q be a q-Sylow subgroup of G. Then Q is nonnormal and does not contain P. Thus G contains at least eight classes of nonnormal subgroups. Since $t(G) = 3$, this is a contradiction.

(ii) G contains no normal p-Sylow subgroup P. In fact, otherwise, by SCHUR's theorem [3, p. 125], there exists a subgroup H of G such that $G = PH$ and $P \cap H = 1$. Then H is insoluble and $t(H) = t(G)-1$. If H is not isomorphic to A_5, then H contains at least $2t(H)+2 = 2t(G)$ classes of nonnormal subgroups. Let us consider the totality of products, each of

[1]) Though TROFIMOV uses the weaker notion "conjugate classes" in his paper [2], his proof remains valid under this stronger notion "isomorphic classes". Further the proof in the present paper does not hold under the notion "conjugate classes". The writer owes this suggestion to Professor RÉDEI and expresses his hearty thanks to him.

which is a product of any one of such subgroups of H with P. Then it occur new $2t(G)$ classes of nonnormal subgroups of G. Since $t(G) \geqq 3$, this is a contradiction. Thus H is isomorphic to A_5. As just above, we have at least fourteen classes of nonnormal subgroups of G. Since $t(G)=4$, this is a contradiction. Thus G has $t(G)$ classes of nonnormal Sylow subgroups.

(iii) Any p-Sylow subgroup P of G is not contained in the centre of its normalizer. In fact, otherwise, by BURNSIDE's theorem [3; p. 133], there exists a normal subgroup H of G such that $G=PH$ and $P \cap H=1$. Then H is insoluble and $t(H)=t(G)-1$. If H is not isomorphic to A_5, then H contains at least $2t(H)+2=2t(G)$ classes of nonnormal subgroups. Further there exist at least two distinct prime factors q, r of the order of H such that the corresponding Sylow subgroups Q, R are nonnormal in H. Let $N(Q)$ and $N(R)$ be the normalizers of Q and R in G respectively. Since H is normal in G, we have $xQx^{-1} \subseteqq H$ for every $x \in G$. Thus, by Sylows's theorem, there exists an element $y \in H$ such that $xQx^{-1}=yQy^{-1}$. Then $y^{-1}x \in N(Q)$. This proves $G=N(Q)H=N(R)H$. By this and the normality of H one may assume that $P \subset N(Q)$. Since Q is the only one q-Sylow subgroup of PQ, if PQ is normal in G, the Q is normal in G. This contradicts either (i) or (ii). Hence the subgroups PQ and PR are nonnormal in G. Thus G contains at least $2t(G)+2$ classes of nonnormal subgroups, which is a contradiction. Thus H is isomorphic to A_5. Since H contains seven isomorphic classes of nonnormal subgroups and three nonnormal Sylow subgroups to the primes 2, 3, 5 we have, as just above, that G contains at least ten classes of nonnormal subgroups. Since $t(G)=4$, this is a contradiction.

(iv) We consider any q-Sylow subgroup Q of G. Now let us assume that Q is abelian. Let $N(Q)$ be the normalizer of Q. Then, from the fact just proved there exists at least one prime factor q' of the order of $N(Q)$ such that for a corresponding Sylow subgroup Q' of $N(Q)$ the product QQ' is not abelian. Clearly QQ' is nonnormal in G. Let us correspond to each prime factor q of the order of G either QQ' or a maximal subgroup Q_1 of Q according as Q is abelian or not. Thus G has again $t(G)$ classes of nonnormal subgroups.

(v) Let there exist a prime factor p of the order of G such that a corresponding Sylow subgroup P is not abelian. Let P_0 be a subgroup of P of order p. Since P_0 is nonnormal, G contains no class of nonnormal subgroups except that of P_0 and the cases mentioned in (ii) and (iv). Therefore for every prime factor q of the order of G, distinct from p, a q-Sylow subgroup Q of G is of order q. If Q is normal in a subgroup H of G then H is in G nonnormal, for otherwise Q were normal in G and this is impossible. Consequently the order of every subgroup QQ' mentioned in (iv) is product of two primes. By virtue of (iii), p is the least prime factor of the order of G.

Assume that G contains two different p-Sylow subgroups, P and P', with $D = P \cap P' \neq 1$. From all such D's we choose a maximal one. The order of the normalizer $N(D)$ of D in G is divisible by $p^2 q$ for some q and $N(D)$ contains D as a characteristic subgroup [3; p. 102]. Since $N(D)$ must be normal in G, so is D normal in G too. This is a contradiction. Therefore $P \cap P' = 1$ for all $P' \neq P$. If the normalizer $N(P)$ of P in G is equal to P, then by a well known theorem of FROBENIUS G contains a normal subgroup H, such that the factor group G/H is isomorphic to P. Hence G is soluble, and this yields a contraction. Therefore $N(P)$ contains P properly. If $N(P)$ is normal in G, then P is also normal in G which is a contradiction. Then $N(P)$ is nonnormal. This is again a contradiction. Thus every p-Sylow subgroup of G is abelian of order at most p^2. Now there exsists just one prime factor p of the order of G such that a corresponding Sylow subgroup is of order p^2. In fact, if there exist no such prime factors, then G is soluble. If there exist two such prime factors, then G contains at least $2t(G)+2$ classes of nonnormal subgroups, which is a contradiction. Further since G is insoluble, p should be equal to two.

(vi) Let M be any maximal subgroup of G. If M is normal in G, then M is of prime index q in G and therefore M is insoluble. Then q is not equal to two. Let Q be a q-Sylow subgroup of G. Then $G = MQ$ and $M \cap Q = 1$. Then Q is contained in the centre of its normalizer, which contradicts (iii). Thus M is nonnormal and therefore M should be conjugate to some QQ' in (iv). Thus any non-maximal subgroup of G is abelian and G is simple. In other words, G is a simple group of Rédei type of even order. Thus by RÉDEI's theorem [1] G is isomorphic to A_5.

This completes the proof.

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