On a new type of radical.[1])

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§ 1. Introduction.

In the theory of noncommutative rings a central role is played by the radical of a ring R. The radical was first defined only in case R satisfied the minimum condition on one-sided ideals,[2]) but later it was extended in different ways to rings without finiteness assumptions.[3]) In such rings most definitions are based on the concept of nilpotency (either for ideals or elements) or quasiregularity. The main purpose of introducing a suitable radical is to obtain some structure theorems for rings having no radical in one sense or another. Hence one is inclined to feel the radical — to speak roughly — a certain measure of "irregularity" of the ring and therefore it is natural to expect that the radical should be zero if the ring is imbeddable in a skewfield, or more generally, if it is free of zerodivisors. Although the radical of N. JACOBSON has been proved to be the most useful radical in the most general case and, besides, the Jacobson radical has also an important group-theoretic interpretation,[4]) it may yet happen that in a domain of integrity, moreover, in a discrete valuation ring, the Jacobson radical does not coincide with the zero ideal.[5]) Therefore, it is justified to say that, in spite of its usefulness, the Jacobson radical is in certain cases superfluously wide.

The present note has for its aim to present a new type of radical, one which contains only zerofactors, but not necessarily exclusively nilpotent elements. We base our definition of radical on the concept of zerofactor, at first introducing a new notion called left- and right-zeroid which is a rather special

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[1]) This paper is an extended version of my previous note [10] published in Hungarian. (Numbers in brackets refer to the Bibliography given at the end of this paper.)

[2]) See KÖTHE [14], DEURING [8], ALBERT [1], VAN DER WAERDEN [21], PERLIS [20], JACOBSON [12], ARTIN—NESBITT—THRALL [2], Чеботарёв [7].

[3]) See BAER [3], JACOBSON [13], BROWN—MCCOY [5].

[4]) See FUCHS [11]. (In rings with one-sided identity the Jacobson radical corresponds to the Frattini subgroup of its additive group considered as an operator-group whose operator-domain is the ring itself.)

[5]) This is the case e. g. in the ring of all p-adic integers.

case of left- resp. right-zerofactor. Our radical will then be defined as the meet of the join of all left-zeroid and the join of all right-zeroid ideals. Many of the main properties of the known radicals retain their validity in the present case, but it will turn out that zerofactors are not so easy to handle, and therefore, our present treatment possesses mainly theoretical rather than practical interest.

After the definition, we shall prove in § 3 that our radical (which may be called the zeroid radical) is the intersection of certain (in general not all) prime ideals of the ring. The next § 4 is devoted to discussing the connections of the new radical with the old types. It will turn out that the zeroid radical in general properly contains the union of all nil ideals[6]) as well as the McCoy radical, but from the point of view of inclusion it has nothing to do with the Jacobson radical. As regards the residue class ring with respect to the radical, it remained an open question whether it is radical free or not; we have proved only that it contains no nonzero nil ideal. In § 6 we show that the minimum condition on one-sided ideals implies that the zeroid radical coincides with the classical one (the join of all nilpotent left ideals). The radical of a matrix ring will also be discussed; under a certain condition it consists of all matrices whose elements lie in the radical of the underlying ring. Finally, some remarks are added concerning the commutative case.

§ 2. Definition.

Let R be an arbitrary associative (but not necessarily commutative) ring. An element a in R is said to be a *left-zerofactor* (*l-zerofactor*) if there is a $b \neq 0$ in R such that $ab = 0$; b is then called a *right-annihilator* of a. If A is a nonvoid subset of R, in particular an ideal[7]) of R, then we call A *l-zerofactor* if each element of A is a l-zerofactor, and call A *annihilable from the right* if for some $c \neq 0$ in R we have $Ac = 0$.

If the ideal A has the property that $A + B$ is a l-zerofactor whenever B is a l-zerofactor ideal, then A will be called a *l-zeroid* (*left-zeroid*) ideal. It is obvious that a l-zeroid ideal is necessarily a l-zerofactor. The existence of l-zeroid ideals is guaranteed by the fact that in any ring the zero ideal 0 is a l-zeroid ideal.

The sum of two l-zeroid ideals is also one. For, suppose A and B are two l-zeroid ideals and C is any l-zerofactor ideal. As B is l-zeroid, $B + C$ is a l-zerofactor and hence, A being l-zeroid, $(A + B) + C = A + (B + C)$ is a l-zerofactor ideal. This proves that $A + B$ is l-zeroid, as stated.

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[6]) An ideal is called a nil ideal if all of its elements are nilpotent. Observe that a nil ideal is not necessarily nilpotent.

[7]) Ideal will throughout mean twosided ideal. For right- resp. leftideal we shall write abbreviatedly r-ideal resp. l-ideal.

Hence it is easy to conclude that the join of all l-zeroid ideals is again a l-zeroid ideal. It will be called the *left-radical* of R and denoted by $Z^{(l)}$.

Changing the roles of left and right, we may introduce analogously the notion of r-zeroid ideals and then define the *right-radical* $Z^{(r)}$ of R as the join of all r-zeroid ideals.

The following example will serve to illustrate that in general the left- and right-radicals are different, moreover, it may happen that one of them properly contains the other. Let R be a ring whose additive group R^+ is a finite abelian group of type $(2, 2)$; a and b will denote the generator elements of the direct summands of R^+ in some direct decomposition. Let the multiplication in R be defined by $ax = x$ and $bx = x$ for all $x \in R$. Then $(a+b)x = 0$ for all $x \in R$. It is readily checked that in R the associative law of multiplication and both distributive laws hold, so that R is a ring of four elements. Now $a+b$ is a left-annihilator of R, i. e. $Z^{(r)} = R$. On the other hand we have $Z^{(l)} = \{0, a+b\}$ ($Z^{(l)}$ is at the same time the maximal nilpotent ideal in R), thus in this example $Z^{(l)} \subset Z^{(r)}$ holds.[8])

In order to obtain a radical which is left-right symmetric, we define the *radical* Z of R as the intersection of its left- and right-radicals:[9])

$$Z = Z^{(l)} \cap Z^{(r)}.$$

Z is the join of all ideals which are both l- and r-zeroid.

It is evident that $Z = R$ if and only if each element of R is both l- and r-zerofactor. Such a ring may be called a *radical ring*.

For the connection of our radical with the known types of radical we refer to § 4.

§ 3. The radical as the intersection of prime ideals.

W. Krull has proved [15] that in a commutative ring the sum of all nilpotent ideals, i. e. the nilpotent radical is the intersection of all prime ideals of the ring. This fact has been proved by McCoy in general rings for the radical introduced by him [18]. A similar result holds for the Jacobson radical in rings with onesided unit element [13]. The theorem we are going to prove shows that these results have an analogue in the case of the zeroid radical; indeed, Z is the intersection of certain prime ideals of the ring.

We call an ideal M *maximal l-zerofactor* (*r-zerofactor*) if it is maximal with respect to the property of being a l-zerofactor (r-zerofactor). Zorn's lemma ensures that every l-zerofactor (r-zerofactor) ideal belongs at least to one maximal l-zerofactor (r-zerofactor) ideal.

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[8]) The sign $\subset$ is used to denote proper inclusion.

[9]) This definition is not entirely the same as given in [10]; there we have understood by the radical what we now call left-radical.

Theorem 1.[10]) *The left-radical $Z^{(l)}$ of R is equal to the intersection of all maximal l-zerofactor ideals M; these M are prime ideals.*

Suppose $Z^{(l)}$ does not belong to some maximal l-zerofactor ideal M. Then $Z^{(l)}+M$ is no l-zerofactor, in violation of the fact that $Z^{(l)}$ is a l-zeroid ideal. Hence $Z^{(l)} \subseteq M$ for all maximal l-zerofactor ideals M.

Conversely, if X is the intersection of all maximal l-zerofactor ideals and A is any l-zerofactor ideal, then some maximal l-zerofactor M contains A, and therefore $X+A \subseteq M$. This establishes that $X+A$ is a l-zerofactor, i. e. X is contained in $Z^{(l)}$, in fact.

What we have still to verify is the primeness of the maximal l-zerofactor ideals M, by a prime ideal being understood an ideal P with the property that the product of two ideals, X and Y, does not belong to P unless either X or Y belongs to P. Now, if neither X nor Y belongs to M, then both $X+M$ and $Y+M$ contain elements $x+m'$ and $y+m''$ $(x \in X, y \in Y, m', m'' \in M)$, respectively, which are not l-zerofactors. If $XY \subseteq M$, then the product $(x+m')(y+m'')=xy+m$ $(m \in M)$ must be a l-zerofactor, say, annihilated by a from the right. Now either a is a right-annihilator of $y+m''$, or, if this is not the case, then $(y+m'')a \neq 0$ is a right-annihilator of $x+m'$. This contradiction establishes the prime character of the maximal l-zerofactor ideals.

Theorem 1 implies at once:

Theorem 1a. *The radical Z of R is the intersection of all maximal l-zerofactor and maximal r-zerofactor ideals which are necessarily prime ideals.*

On account of the fact that the residue class ring with respect to a prime ideal does not contain annihilable ideals, it follows from a general structure theorem of Birkhoff:[11])

Theorem 2. *The residue class ring R/Z of R with respect to the radical is a subdirect sum of rings without annihilable ideals.*

Calling a ring *semisimple* if its radical Z is 0, we have

Corollary. *A semisimple ring is a subdirect sum of rings without annihilable ideals.*

§ 4. Connections between the different radicals.

In this section the following known radicals will be considered: *the nilpotent radical N* (the join of all nilpotent ideals), *the nil radical U* (or *the upper radical* defined as the join of all nil ideals of the ring), *the McCoy*

[10]) In the commutative case this theorem may be found in [9], and in the noncommutative case a similar result is contained in Curtis' paper [6].

[11]) See Birkhoff [4], McCoy [17] and Brown—McCoy [5].

radical M (the set of all elements belonging to no m-system[12]) not containing 0) and *the Jacobson radical* J (the join of all right-quasiregular r-ideals). Before entering into the discussion of the connection of the zeroid radical with the mentioned radicals, we observe that in general rings the following inclusion relations are valid:

$$N \subseteq M \subseteq U \subseteq J. \tag{1}$$

Indeed, the inclusion $N \subseteq M$ follows at once from the fact that M is the intersection of all prime ideals P of the ring,[13]) and therefore $A^s = 0 \subseteq P$ implies $A \subseteq P$, i. e. all nilpotent ideals are contained in M. To prove the second inclusion, observe that if for some $a \in R$ we have $a^n \neq 0$ $(n = 1, 2, \ldots)$, then a maximal ideal P containing no power of a is prime. For, each proper overideal of P contains some power of a, and therefore the product of two such overideals[14]) is never contained in P. Consequently, only nilpotent elements may belong to $M = \bigcap_{\text{all primes } P} P$, and hence $M \subseteq U$.[15]) For the last inclusion of (1) we refer to JACOBSON's paper [13] where it is shown that each nilpotent element is right-quasiregular.

As regards the zeroid radical Z, from Theorem la it results immediately: $M \subseteq Z$. Moreover, we may prove the inclusion relation $U \subseteq Z$. To this end, let h be a nilpotent element and A a l-zerofactor ideal. If $h^n = 0$, then $a' = (h + a)^n$ $(a \in A)$ belongs to A, because each term in the expansion is either 0 or contains a as a factor. Hence some $b \neq 0$ annihilates a' from the right. Let l be the least exponent for which b is a right-annihilator of $(h + a)^l$. Then[16]) $(h + a)^{l-1} b \neq 0$ is a right-annihilator of $h + a$, i. e. $h + a$ is a l-zerofactor. Consequently, every nil ideal is l-zeroid, and similarly, r-zeroid. Hence $U \subseteq Z$, in fact.

A simple example will show that in general Z does not coincide with U, not even under the assumption of commutativity and maximal condition. For instance, let P denote the ring of all polynomials in two indeterminates u and v, with rational numbers for coefficients, and let R be the residue class ring $P/(u^2, uv)$ of P with respect to the ideal (u^2, uv). Then it is easy to see that $(u)/(u^2, uv)$ is the join of all nil ideals (= nilpotent ideals), while $(u, v)/(u^2, uv)$ is the zeroid radical of this ring.

In order to make clear that the direction in which the nil radical was extended by N. JACOBSON to his radical is quite different from ours, we show by examples that it may well happen that the zeroid radical Z properly con-

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12) By an m-system S is meant a subset of the ring R with the property: $a, b \in S$ imply the existence of an $x \in R$ such that $axb \in S$.

13) For this result see McCoy [18].

14) Clearly, there is no loss of generality in confining ourselves to the overideals of P.

15) By LEVITZKI's result [16], M is the lower radical L of the ring in the sense of BAER [3] and since $L \subseteq U$ holds, the relation $M \subseteq U$ follows immediately from these well-known results. But it may easily be proved directly as shown in the text.

16) In case $l = 1$, put here simply b.

tains the Jacobson radical J as well as conversely, and in the most general case neither contains the other. For the possibility $J \subset Z$ the last example will serve where $J = (u)/(u^2, uv)$. To illustrate the case $Z \subset J$, let us consider the ring of rational numbers with odd denominators; a simple calculation shows that $J = (2)$, while Z is obviously 0.

Before illustrating the most general case, we prove a lemma which has its own interest too.

Lemma 1.[17]) *If R is the direct sum of a finite number of nonzero rings R_i (which are ideals in R),*

$$R = R_1 + \ldots + R_n, \tag{2}$$

and $Z_i^{(l)}$ ($Z_i^{(r)}$) is the l-radical (r-radical) of R_i, then for the l-radical $Z^{(l)}$ of R we have

$$Z^{(l)} = \begin{cases} R \text{ if } Z_i^{(l)} = R_i \text{ for some } i = 1, \ldots, n, \\ Z_1^{(l)} + \ldots + Z_n^{(l)} \text{ if } Z_i^{(l)} \neq R_i \text{ for all } i = 1, \ldots, n, \end{cases}$$

and the same for the r-radical $Z^{(r)}$ of R.

It suffices to verify the statement for the l-radical $Z^{(l)}$. It is immediately seen that an element $a = a_1 + \ldots + a_n$ of R ($a_i \in R_i$) is a l-zerofactor if and only if for at least one i the component a_i is a l-zerofactor in R_i. Hence it results that if $Z_i^{(l)} = R_i$ for some i, then every element of R is a l-zerofactor, i. e. $Z^{(l)} = R$. But if $Z_i^{(l)} \neq R_i$ for every i, then taking any l-zerofactor ideal B_i in each R_i, we see that if an ideal A is l-zeroid, then $A + B_i^*$ (where $B_i^* = R_1 + \ldots + R_{i-1} + B_i + R_{i+1} + \ldots + R_n$) must be a l-zerofactor from which we infer that $A_i + B_i$ (A_i the ith component of A in decomposition (2)) is a l-zerofactor, that is to say, A_i is a l-zeroid ideal in R_i. This implies the second alternative of the statement.

We also remark that if R is the (discrete) direct sum of an infinity of its subrings then R is a radical ring. In fact, in this case every element of R is a l-zerofactor as well as a r-zerofactor.

Now, from the lemma and the fact that the Jacobson radical of a direct sum of rings is the direct sum of the respective Jacobson radicals, we conclude that the direct sum of the two rings given as examples in the paragraph last but one before Lemma 1 is an instance for a ring in which none of the Jacobson radical and the zeroid radical contains the other.

What has been said about the connections of the different types of radical implies

Theorem 3. *In general rings for the different radicals the following situation holds:*

$$N \subseteq M \subseteq U \subseteq \begin{cases} J \\ Z. \end{cases}$$

[17]) Observe that Lemma 1 is not true for the radical Z in place of the l-radical.

§ 5. The residue class ring with respect to the radical.

The known types of radical have the property that the residue class ring with respect to the radical has zero radical where, obviously, both radicals are to be taken in one and the same sense. Whether or not the same result holds for the zeroid radical is an open question. Here we prove the following weaker result.

Theorem 4. *The nil radical of the residue class ring R/Z with respect to the zeroid radical Z of R is always zero.*

We show that if some ideal C/Z in R/Z is a nil ideal then $C/Z = Z/Z$. Let $c \in C$ and $c^k \in Z$. If A is a l-zerofactor ideal in R and $a \in A$, then $(c+a)^k \in Z+A$, i. e. $(c+a)^k$ and thus also $c+a$ is a l-zerofactor. Consequently, C is a l-zeroid ideal and analogously, a r-zeroid ideal, completing the proof.

§ 6. The radical of a ring with minimum condition on one-sided ideals.

Assume the ring R contains a unit element e and the r-ideals of R satisfy the minimum condition. We shall prove that in this case the zeroid radical Z contracts to the classical radical N, i. e. the join of all nilpotent r-ideals.

Before entering into the proof of this statement, let us remark that without the existence of a unit element e this assertion need not be true.[18]) If R_1, R_2 are simple rings (with minimum condition on r-ideals) such that $R_1^2 = 0$ and $R_2^2 = R_2$, then the nilpotent radical of $R_1 + R_2$ (direct sum) is R_1, while the zeroid radical coincides with the whole ring, in view of Lemma 1.

Recall that in a ring with minimum condition on r-ideals the nilpotent radical and the nil radical coincide, so that they are equal to the intersection of all prime ideals P of the ring.[19]) Therefore, if we can show that each prime ideal P is a l-zerofactor, then this will imply $Z \subseteq Z^{(l)} = \cap P = N$ (use Theorem 1) whence by Theorem 3 we shall obtain $Z = N$ and this will establish our assertion.

Let $P (\neq R)$ be a (prime) ideal in R. If P were not a l-zerofactor, then, by the minimum condition on r-ideals, there would be a minimal r-ideal Q_r which is contained in P and is not a l-zerofactor. Let c denote an element in Q_r which is not a l-zerofactor. Then, by minimality, we must have $Q_r = (c)_r$, the r-ideal generated by c. Further, $c(c)_r$ is a r-ideal in Q_r and, since it con-

[18]) From the discussions of this section it will be clear that instead of assuming the existence of an identity element it will suffice to suppose the presence of a right identity in the ring.

[19]) Cf. Theorem 3.

tains c^2, it is neither a l-zerofactor; consequently, we have

$$c(c)_r = (c)_r.$$

This equality ensures the existence of an element $q \in (c)_r = Q_r$ such that $cq = c$, that is, $c(q-e) = 0$. The last equation shows that c is a l-zerofactor, for q as an element of P is surely different from e. The contradiction completes the proof of

Theorem 5. *In a ring with unit element and minimum condition on r-ideals the radical Z coincides with the join of all nilpotent r-ideals.*

In view of this theorem we see that a ring with unit element is semi-simple in the classical sense (i. e. contains no nilpotent r-ideals other than 0 and satisfies the minimum condition on r-ideals) if and only if it contains no zeroid ideals different from 0 and satisfies the minimum condition on one-sided ideals.

Our last proposition may be generalized by demonstrating that *a ring which is regular in the sense of* J. v. NEUMANN [19] *has zero zeroid radical*, i. e. $Z = 0$. The proof is carried out by showing at first that each (prime) ideal P of a regular ring R is a l-zerofactor. If $a \in P$ then there is an $x \in R$ such that $axa = a$. Since $P \neq R$, we have $xa \neq e$ (the unity of R); consequently, $a(xa-e) = 0$, i. e. a is a l-zerofactor and hence $Z = M$ (the McCoy radical). By making use of Theorem 3, the proof will be completed by observing that the nil radical U of a regular ring is necessarily 0, for $(a)_r \neq 0$ contains the idempotent element ax.

§ 7. The radical of a matrix ring.

We remember that the ring of all $n \times n$ matrices over a ring R possesses the property that its nilpotent radical arises as the ideal of all matrices whose elements lie in the nilpotent radical of R. We next intend to show that the corresponding result for the zeroid radical can also be proved provided we make a further assumption on R (see (*) below).

For convenience, we introduce the following notations. If S is any ring, the complete matrix ring of order n over S will be denoted by S_n and the zeroid radical of S_n by $Z(S_n)$. It is readily seen that if A is an ideal in S then A_n is an ideal in S_n.

We suppose the ring R under consideration satisfies:

(*) *If A is a l-zerofactor (r-zerofactor) ideal and $a_1, \ldots, a_m$ is a finite subset of A, then there is an element $c \neq 0$ in R with the property $a_i c = 0$ (resp. $c a_i = 0$) for $i = 1, \ldots, m$.*

In order to verify that under (*) we have $Z_n = Z(R_n)$, we first prove a simple lemma.

Lemma 2. *An ideal Ξ in the complete matrix ring R_n over R with property (*) is a l-zerofactor if and only if there exists a l-zerofactor ideal A in R such that $\Xi \subseteq A_n$.*

At first, in order to verify the sufficiency of the stated condition, suppose A is a l-zerofactor ideal and $\Xi \subseteq A_n$. If α is a matrix in Ξ, then the set of all elements of α is a finite set in A, consequently, by (*), there exists an element $c \neq 0$ in R which is a common right annihilator of this set. But then the diagonal matrix $\langle c, \ldots, c \rangle$ of order n annihilates α from the right.

We assume, conversely, that Ξ is a l-zerofactor ideal in R_n. Let A be the ideal in R generated by all the elements of the matrices in Ξ. Plainly, $\Xi \subseteq A_n$ and it is enough to show that A is a l-zerofactor. Let $a^{(\nu)} \in A$, so that $a^{(\nu)} = a_1^{(\nu)} + \cdots + a_{m_\nu}^{(\nu)}$ where, without loss of generality, we may suppose that $a_i^{(\nu)}$ is, say, an element standing in the $(j_{i\nu}, k_{i\nu})$ position of a matrix $\alpha_i^{(\nu)} \in \Xi$. If we denote by $(x)_{jk}$ the matrix with x in the (j, k) position and zeros elsewhere, then the matrices[20])

$$\beta_i^{(\nu)} = \sum_{s=1}^{n} (x_\nu)_{sj_{i\nu}} \alpha_i^{(\nu)} (y_\nu)_{k_{i\nu}s}$$

are readily seen to be diagonal matrices of the type $\langle x_\nu a_i^{(\nu)} y_\nu, \ldots, x_\nu a_i^{(\nu)} y_\nu \rangle$. All of $\beta_1^{(\nu)}, \ldots, \beta_{m_\nu}^{(\nu)}$ belong to Ξ, so that the same is true for

$$\beta^{(\nu)} = \beta_1^{(\nu)} + \cdots + \beta_{m_\nu}^{(\nu)} = \langle x_\nu a^{(\nu)} y_\nu, \ldots, x_\nu a^{(\nu)} y_\nu \rangle$$

and also for $\beta = \beta^{(1)} + \cdots + \beta^{(t)}$, i. e. β has a right annihilator matrix $\gamma \neq 0$ in R_n. Now, any nonzero element c of γ annihilates $\sum_{\nu=1}^{t} x_\nu a^{(\nu)} y_\nu$ from the right, showing that the ideal RAR is a l-zerofactor. We infer that $A^3 (\subseteq RAR)$ and hence A is a l-zerofactor ideal in R. This completes the proof of Lemma 2.

As an immediate consequence of this lemma we obtain that *A_n is a l-zerofactor ideal in R_n if and only if A is a l-zerofactor ideal in R.* This observation, together with the same on r-zerofactors, is important in the demonstration of

Theorem 6. *If R satisfies (*), then the radical of the ring of all matrices with elements in R consists of all matrices whose elements lie in the radical of R.*

Let $\Xi = Z(R_n)$ and A a l-zerofactor ideal in R. Thus A_n is a l-zerofactor ideal in R_n. As $\Xi + A_n$ must be a l-zerofactor ideal, by Lemma 2 there is some l-zerofactor ideal B in R such that $\Xi + A_n \subseteq B_n$. Since A was arbitrary, we conclude that $\Xi \subseteq Z_n^{(l)}$. By symmetry we have $\Xi \subseteq Z_n^{(r)}$ whence $\Xi \subseteq Z_n$.

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[20]) x_ν and y_μ denote arbitrary elements of R.

On the other hand, if H is any l-zerofactor ideal in R_n, then by Lemma 2 we have $H \subseteq B_n$ for some l-zerofactor ideal B of R. Thus $Z_n + H \subseteq Z_n + B_n = (Z+B)_n$. Since $Z+B$ must be a l-zerofactor, we are led to the conclusion that $Z_n + H$ is a l-zerofactor ideal in R_n, i. e. $Z_n \subseteq Z^{(l)}(R_n)$. Similarly we have $Z_n \subseteq Z^{(r)}(R_n)$ and the theorem is proved.

§ 8. Remarks concerning commutative rings.

1. N. JACOBSON has proved that in an algebra over a field Φ the elements of his radical are either nilpotent or are transcendental over Φ. A similar result may be established for our radical in commutative algebras $\mathfrak{A}$ provided $\mathfrak{A}$ satisfies the trivial necessary condition of containing at least one regular (i. e. no zerofactor) element.

Theorem 7. *Let $\mathfrak{A}$ be a commutative algebra over a field Φ, with at least one regular element. Then besides the nilpotent elements only transcendental elements over Φ may belong to the radical Z of $\mathfrak{A}$.*

For, let a be algebraic over the underlying field Φ. Then the subalgebra $\mathfrak{B}$ generated by a has a finite basis over Φ and it follows at once the existence of an integer n such that $\mathfrak{B}a^n = \mathfrak{B}a^{n-1}$. Assume $a \in Z$, the radical of $\mathfrak{A}$, and $a^m \neq 0$ for each positive integer m. Then there is a y in $\mathfrak{B}$, and so in Z, satisfying $ya^n = qa^n$ with a regular element $q \in \mathfrak{A}$. Thus $(y-q)a^n = 0$, i. e. the ideal $(y-q)$ is a zerofactor. Since $(y) \in Z$, we obtain that the ideal $(y)+(y-q)$ is a zerofactor, which is absurd, q being a regular element belonging to it. This also shows that *if $\mathfrak{A}$ is algebraic, the radical of $\mathfrak{A}$ is the totality of the nilpotent elements.*

2. It is a well known fact that every ring can be represented as a subdirect sum of subdirectly irreducible rings.[21]) Therefore it might be of some interest to have information about the radical of a subdirectly irreducible ring. The result we find in the commutative case will show that the radical of such a ring has a very simple structure.

Theorem 8. *The radical of a subdirectly irreducible commutative ring consists of the set of all zerofactors.*

It suffices to prove that if both x and y are zerofactors, then the ideal $(x)+(y)$ is a zerofactor. But

$$C = 0:[(x)+(y)] = 0:x \cap 0:y \neq 0,$$

since $0:x$ and $0:y$ are ideals and the ring is by hypothesis subdirectly irreducible. Any nonzero element of C annihilates each element of $(x)+(y)$.

[21]) See McCoy [17], for instance.

It is easy to prove that the radical Z has now a nonzero annihilator. In fact, let y be any nonzero element of the intersection of all ideals $0:x$ where x runs over all elements of Z; then $Zy=0$. It also follows that Z is a prime ideal; indeed, it is the only maximal zerofactor ideal which is prime by Theorem 1.

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(Received October 26, 1954.)