On a class of infinite products whose value can be expressed in closed form.

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1. The familiar formula of WALLIS

$$(1)\qquad \prod_{m=1}^{\infty} \frac{(2m)^2}{(2m-1)(2m+1)} = \frac{\pi}{2}$$

gives a very simple, not-trivial example of an infinite product with well-known value. L. FEJÉR suggested the problem to find a possibly wide class of infinite products, including the WALLIS product, whose value can be expressed in finite form by means of the elementary functions.

In a previous paper[1]) I discussed the product

$$(2)\qquad P = \prod_{n=0}^{\infty} \frac{\left(\dfrac{a_n + a_{n+1} + \cdots + a_{n+\nu}}{\nu+1}\right)^{\nu+1}}{a_n a_{n+1} \cdots a_{n+\nu}}$$

for fixed $\nu \geqq 1$, $\{a_n\}$ meaning a strictly increasing sequence of *positive* numbers; it was proved that P is convergent if and only if the series $\sum_{n=0}^{\infty}\left(\frac{a_{n+\nu}}{a_n}-1\right)^2$ converges, and some formulae were deduced for the case of an arithmetical progression.

2. Now let d, A and D mean fixed *complex* numbers $\neq 0$, and let z be a complex variable. We denote by $\mathfrak{A}_\nu(z)$, $\mathfrak{G}_\nu(z)$ the arithmetic and geometric mean[2]), respectively, of $z, z+d, \ldots, z+(\nu-1)d$, and consider, for a fixed $\nu \geqq 2$, the product

$$(3)\qquad Q = \prod_{\substack{z=A+nD \\ n=0,1,2,\ldots}} \left(\frac{\mathfrak{A}_\nu(z)}{\mathfrak{G}_\nu(z)}\right)^{\nu} = \prod_{\substack{z=A+nD \\ n=0,1,2,\ldots}} \frac{\left(z+\dfrac{\nu-1}{2}d\right)^{\nu}}{z(z+d)\ldots(z+\overline{\nu-1}\,d)};$$

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[1]) Cf. M. MIKOLÁS, Sur un produit infini, *these Acta*, **12 A** (1950), 68—72. — The auxiliary function used here, $\mathfrak{S}(t,\alpha) = \prod_{n=0}^{\infty}\left(1-\frac{t^2}{(n+\alpha)^2}\right)$, may be written also in the form $\Gamma(\alpha)^2/\Gamma(\alpha+t)\Gamma(\alpha-t)$.

[2]) Any value of the νth root may be chosen.

Q may be regarded plainly as a generalization of the WALLIS product. — Let, for the sake of brevity, $\alpha=\frac{A}{D}$, $\delta=\frac{d}{D}$, $h=\frac{\nu-1}{2}$.

We need the following well-known facts from the theory of the gamma-function:

(4) $$\Gamma(z)=\lim_{n\to\infty}\frac{n!\,n^z}{z(z+1)\dots(z+n)} \qquad (z\neq 0,-1,-2,\dots),$$

(5) $$\Gamma(z+1)=z\Gamma(z) \qquad (z\neq 0,-1,-2,\dots),$$

(6) $$\Gamma(z)\Gamma(1-z)=\pi\operatorname{cosec}\pi z \qquad (z\neq 0,\pm 1,\pm 2,\dots),$$

(7) $$\prod_{\mu=0}^{\nu-1}\Gamma\left(z+\frac{\mu}{\nu}\right)=(2\pi)^h\nu^{\frac{1}{2}-\nu z}\Gamma(\nu z) \qquad (\nu z\neq 0,-1,-2,\dots),$$

furthermore

(8) $$\Gamma(m)=(m-1)! \qquad (m=1,2,\dots),$$

(9) $$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

Theorem I. *The product* (3) *converges if and only if neither of the numbers* $\alpha+h\delta$, $\alpha+\mu\delta$ ($\mu=0,1,\dots,\nu-1$) *is* 0 *or a negative integer. In this case Q can be written in closed form by means of the gamma-function, namely*

(10) $$Q=\Gamma(\alpha)\Gamma(\alpha+\delta)\cdots\Gamma(\alpha+\overline{\nu-1}\,\delta)\cdot\Gamma(\alpha+h\delta)^{-\nu}.$$

Especially, if $\delta=1$, *we have for* $\nu=2$

(11) $$Q=\alpha\left(\frac{\Gamma(\alpha)}{\Gamma\left(\alpha+\frac{1}{2}\right)}\right)^2$$

and for any $\nu>2$

(12) $$Q=\frac{(\alpha+\nu-2)(\alpha+\nu-3)^2\cdots(\alpha+[h])^{2h-[h]}}{\alpha(\alpha+1)^2\cdots(\alpha+[h]-1)^{[h]}}\left(\frac{\Gamma(\alpha+[h])}{\Gamma(\alpha+h)}\right)^{\nu};$$ [3])

if $\delta=\frac{1}{\nu}$, (10) *becomes*

(13) $$Q=(2\pi)^h\nu^{\frac{1}{2}-\nu\alpha}\frac{\Gamma(\nu\alpha)}{\Gamma\left(\alpha+\frac{h}{\nu}\right)^{\nu}};$$

in case $\alpha+h\delta=1$ *we obtain by putting* $\Theta_\mu=1-(\alpha+\mu\delta)$

(14) $$Q=\prod_{\mu=0}^{\left[\frac{\nu}{2}\right]-1}\frac{\pi\Theta_\mu}{\sin\pi\Theta_\mu},$$

provided that neither of $\Theta_1,\Theta_2,\dots$ *is an integer.*

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[3]) $[h]$ denotes the integer part of h. — If ν is odd, the last factor in (12) (including gamma-values) may be plainly omitted.

Proof. 1° If one of $\alpha+h\delta, \alpha+\mu\delta$ $(\mu=0,1,\ldots,\nu-1)$ is zero or a negative integer, then the nominator or denominator of

$$(15)\qquad Q_N=\prod_{n=0}^{N}\frac{(A+nD+hd)^{\nu}}{(A+nD)(A+nD+d)\cdots(A+nD+\overline{\nu-1}d)}=$$
$$=\prod_{n=0}^{N}\frac{(\alpha+h\delta+n)^{\nu}}{(\alpha+n)(\alpha+\delta+n)\cdots(\alpha+\overline{\nu-1}\delta+n)}$$

vanishes for N sufficiently large, and so Q_N has the value 0, or it has no meaning, respectively.

Otherwise we write

$$Q_N=\frac{\{(\alpha+h\delta)(\alpha+h\delta+1)\cdots(\alpha+h\delta+N)\}^{\nu}}{\alpha(\alpha+1)\cdots(\alpha+N)\cdot(\alpha+\delta)(\alpha+\delta+1)\cdots(\alpha+\delta+N)\cdots(\alpha+\overline{\nu-1}\delta)\cdots(\alpha+\overline{\nu-1}\delta+N)}=$$
$$=\left(\frac{(\alpha+h\delta)(\alpha+h\delta+1)\cdots(\alpha+h\delta+N)}{N!\,N^{\alpha+h\delta}}\right)^{\nu}\prod_{\mu=0}^{\nu-1}\frac{N!\,N^{\alpha+\mu\delta}}{(\alpha+\mu\delta)(\alpha+\mu\delta+1)\cdots(\alpha+\mu\delta+N)}.$$

Here, by (4), every fraction has a limit as $N\to\infty$, moreover

$$(16)\qquad Q=\prod_{n=0}^{\infty}\frac{(\alpha+h\delta+n)^{\nu}}{(\alpha+n)(\alpha+\delta+n)\cdots(\alpha+\overline{\nu-1}\delta+n)}=\Gamma(\alpha+h\delta)^{-\nu}\prod_{\mu=0}^{\nu-1}\Gamma(\alpha+\mu\delta).$$

2° Let $\delta=1$. For $\nu=2$ the last formula becomes at once (11) because of (5). — If $\nu>2$, we use the relation

$$(17)\qquad \Gamma(z)=(z-1)(z-2)\cdots(z-\lambda)\Gamma(z-\lambda)\qquad(\lambda=1,2,\ldots),$$

arising from (5) by repetition; considering that

$$\Gamma(\alpha+[h]+p)=(\alpha+[h]-1)\cdots(\alpha+[h])\Gamma(\alpha+[h])\qquad(p=1,2,\ldots,2h-[h]),$$
$$\Gamma(\alpha+[h]-q)=\{(\alpha+[h]-q)\cdots(\alpha+[h]-1)\}^{-1}\Gamma(\alpha+[h])\qquad(q=1,2,\ldots,[h]),$$

(16) may be written in the form (12).

Concerning (13), (14), we have only to put $\delta=\frac{1}{\nu}$, $\delta=\frac{1-\alpha}{h}$ in (16) and then to apply the multiplication theorem of Gauss (7), furthermore the functional equations (5), (6), respectively.

3. In some particular cases it is possible to find for Q a finite expression which is *free* from gamma-values. We have namely the following

Theorem II. *Assume that neither of $\alpha+h\delta$, $\alpha+\mu\delta$ $(\mu=0,1,\ldots,\nu-1)$ is 0 or a negative integer. — On the basis of (5)—(9) exclusively, Q can be transformed into a closed analytical expression built from $2,\pi,\nu$, factorial numbers, and from α, δ by means of rational operations, square roots and the sine-function, if and only if at least one of the following conditions is satisfied: 1) δ is an integer and $(\nu-1)\delta$ is even, 2) $2\alpha+(\nu-1)\delta=K$, where K means an integer different from zero and the negative even numbers.*

Proof. Let $\alpha, \alpha+\delta, \ldots, \alpha+(\nu-1)\delta, \alpha+h\delta$ be complex numbers, different from $0, -1, -2, \ldots$.

1° Suppose that δ and $h\delta$ are integers (i. e. $(\nu-1)\delta \equiv 0 \pmod 2$). Then it follows by (17) $(\mu=0,1,\ldots,\nu-1)$

$$(18)\quad \Gamma(\alpha+\mu\delta)=\begin{cases}(\alpha+\mu\delta-1)(\alpha+\mu\delta-2)\cdots(\alpha+h\delta)\Gamma(\alpha+h\delta) & \text{if } (\mu-h)\delta>0,\\ \{(\alpha+\mu\delta)(\alpha+\mu\delta+1)\cdots(\alpha+h\delta-1)\}^{-1}\Gamma(\alpha+h\delta) & \text{if } (\mu-h)\delta<0,\end{cases}$$

so that we obtain from (10) by substitution and simplification with $\Gamma(\alpha+h\delta)^{\nu}$ a finite and in α, δ rational expression for Q.

Let $2\alpha+(\nu-1)\delta=K$ $(K=1,2,3,\ldots; -1,-3,-5,\ldots)$. This implies (cf. (5), (9))

$$(19)\quad \Gamma(\alpha+h\delta)=\Gamma\left(\frac{K}{2}\right)=\begin{cases}(s-1)! & \text{for } K=2s \quad (s=1,2,\ldots),\\ 2^{-s}(2s+1)!!\sqrt{\pi} & \text{if } K=2s+1 \quad (s=0,1,2,\ldots),\\ (-2)^{s+1}\dfrac{\sqrt{\pi}}{(2s+1)!!} & \text{if } K=-(2s+1) \; (s=0,1,2,\ldots).\end{cases}$$

Next consider the product

$$(20)\qquad \Gamma(\alpha+\mu\delta)\Gamma(\alpha+\overline{\nu-\mu-1}\,\delta)=\Gamma(\alpha+\mu\delta)\Gamma(K-\alpha-\mu\delta)$$

with an integer μ, $0\leqq\mu\leqq\nu-1$; the second term contains two factorial numbers if $\alpha+\mu\delta$ is a positive integer (cf. (8)), otherwise it can be represented, on the basis of (17) and (6), as a closed expression of π, α, δ by means of rational operations and sine-values.

2° Now, we should like to know *all* the cases, in which the right-hand side of (10) can be written by using (5)—(9) in form required above. As it is at once to see, in any case in question $\Gamma(\alpha+h\delta)^{-\nu}\prod_{\mu=0}^{\nu-1}\Gamma(\alpha+\mu\delta)$ must be reducible by (5), (6), (7) (in a definite number of steps) so that the closed expression obtained does not contain values of $\Gamma(z)$ except possibly those with $z=1,2,3,\ldots; \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \ldots$.

Concerning the factor $\Gamma(\alpha+h\delta)$, this implies plainly two possibilities: 1) it occurs only 'apparently', i. e. we can simplify with $\Gamma(\alpha+h\delta)^{\nu}$ (after transformations permitted) in the product mentioned; 2) $\alpha+h\delta$ is a positive integer or one of the fractions $\pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$, i. e.

$$\alpha+h\delta=\frac{K}{2}\qquad (K=1,2,3,\ldots; -1,-3,-5,\ldots).$$

The case 1) can be realised only if any of the values $\Gamma(\alpha+\mu\delta)$ $(\mu=0,1,\ldots,\nu-1)$ can be written by (5) and (17), respectively, as a product of (18) type; but such a relation between $\Gamma(\alpha+\mu\delta)$ and $\Gamma(\alpha+h\delta)$ assumes that $(\mu-h)\delta$, and therefore, in particular, $(\mu+1-h)\delta-(\mu-h)\delta=$

$=\delta$ and $(\nu-1-h)\delta=\frac{1}{2}(\nu-1)\delta$ are integers $(\mu=0,1,\ldots,\nu-1)$. Thus we have got the first condition of the theorem.

In the case 2) one has $2\alpha+(\nu-1)\delta=K$ $(K=1,2,3,\ldots;-1,-3,-5,\ldots)$, i. e. the second condition must be fulfilled.

This completes the proof.

4. We give a few examples.

(11) becomes for $\nu=2$ and $\alpha=\delta=1$

$$\prod_{m=1}^{\infty}\frac{\left(m+\frac{1}{2}\right)^2}{m(m+1)}=\frac{4}{\pi}, \tag{21}$$

and with $\alpha=\frac{1}{2}$, $\delta=1$ the formula (1) of WALLIS.

From (13) it results by putting $\alpha=\frac{1}{2\nu}$

$$\prod_{n=0}^{\infty}\frac{[(2n+1)\nu]^{\nu}}{(2n\nu+1)(2n\nu+3)\cdots(2n\nu+2\nu-1)}=2^{\frac{\nu-1}{2}}\quad(\nu=2,3,\ldots), \tag{22}$$

while for $\alpha=\frac{1}{2}+\frac{1}{2\nu}$ we obtain

$$\prod_{n=0}^{\infty}\frac{[2(n+1)\nu]^{\nu}}{[(2n+1)\nu+1][(2n+1)\nu+3]\cdots[(2n+1)\nu+2\nu-1]}=(2\pi)^{h}\nu^{-\frac{\nu}{2}}\Gamma\left(\frac{\nu+1}{2}\right)=$$

$$=\begin{cases}\dfrac{(2\pi)^h h!}{\nu^{\frac{\nu}{2}}} & \text{for } \nu=3,5,7,\ldots,\\[2ex] \dfrac{1}{\sqrt{2}}\left(\dfrac{\pi}{\nu}\right)^{\frac{\nu}{2}}(\nu-1)!! & \text{for } \nu=2,4,6,\ldots.\end{cases} \tag{23}$$

Since we have, by (6) and (7),

$$\prod_{l=1}^{\nu-1}\sin\frac{l\pi}{\nu}=\frac{\nu}{2^{\nu-1}}\quad(\nu=2,3,\ldots), \tag{24}$$

(14) transforms itself for $\nu=2\varrho$, $\alpha=\frac{1}{2\varrho}$, $\delta=\frac{1}{\varrho}$ $(\varrho=1,2,\ldots)$ into

$$\prod_{n=0}^{\infty}\frac{[(2n+1)\varrho]^{2\varrho}}{(2n\varrho+1)(2n\varrho+3)\cdots(2n\varrho+4\varrho-1)}=\frac{1}{2}\left(\frac{\pi}{\varrho}\right)^{\varrho}(2\varrho-1)!!, \tag{25}$$

which is a remarkably simple generalization of (1) $(\varrho=1)$.

It may be mentioned that (25) follows easily also from (22) and (23) if we take $\nu=2\varrho$ and multiply the corresponding terms.

(Received January 21, 1955.)