Generalization of a theorem of Birkhoff concerning maximal chains of a certain type of lattices.

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To Professor L. Kalmár on his 50th birthday.

Let L be any lattice and let a, b be any pair of its elements such that $a < b$. Then the set of all elements x of L such that $a \leqq x \leqq b$ is a sublattice of L and is called the *closed interval* $[a, b]$. If the inequalities $a < x < b$ are not satisfied by any x in L (i. e., if $[a, b]$ consists only of the elements a and b), then we say "a is covered by b" or "b covers a" and we write $a \prec b$ or $b \succ a$.

As usual, a finite chain

$$C: \; a_0 < a_1 < \cdots < a_m \qquad (m \text{ finite})$$

of elements of L is called *maximal* (and *of length* m) if $a_i \prec a_{i+1}$ $(i = 0, 1, \ldots, m-1)$. But in this paper we shall use the term "maximal" also for chains of infinite length in the following generalized sense: *A chain C of some elements of the lattice L is called maximal if it is not a proper subchain of any chain C′ in L.* Clearly, for finite chains our generalized definition is equivalent to the usual one. By *the length of an infinite maximal chain* we mean the set-theoretical power of the set of its elements.

The problem of this paper is an extension of one considered by DEDEKIND [**3**, p, 397], BIRKHOFF [**1**, p. 66] and also previously by the author [**4**, p. 240]. We recall these results in a modified and somewhat generalized form.

BIRKHOFF has shown, on basis of the investigations of DEDEKIND, the following important theorem:

Theorem 1. *Let* $[a, b]$ *be any closed interval of a lattice in which the assumptions are satisfied:*

(A_1) $x \neq y$ *and* $x, y \succ u$ $(x, y, u \in [a, b])$ *imply* $x \cup y \succ x, y$;

(B_1) *all chains in* $[a, b]$ *are finite.*

Then

(S) *all maximal chains between a and b have the same length.*

It is known [**1**, p. 100] that for lattices of finite length $(\mathbf{A}_1)$ is equivalent to

$$(\mathbf{A}_2)\ x \cap y \prec y \quad \textit{implies} \quad x \prec x \cup y \quad (x, y \in [a, b]),$$

and for lattices of infinite length it is a consequence of $(\mathbf{A}_2)$. Since, by $(\mathbf{B}_1)$, the interval $[a, b]$ is a sublattice of finite length in L, it follows that in Theorem 1, $(\mathbf{A}_1)$ may be replaced by $(\mathbf{A}_2)$.

Two years ago, the author has shown in his above-mentioned paper that in addition to $(\mathbf{A}_2)$ it suffices to assume the following condition which is considerably weaker than $(\mathbf{B}_1)$:

$(\mathbf{B}_2)$ *there exists a finite maximal chain between a and b.*

In fact, the author has proved

Theorem 2. *If a closed interval $[a, b]$ of a lattice L satisfies $(\mathbf{A}_2)$ and $(\mathbf{B}_2)$, then the length of any chain between a and b does not exceed the length of the finite maximal chain of $(\mathbf{B}_2)$.*

Consequently, statement (**S**) also holds in $[a, b]$.

Now, our purpose is to discuss the maximal chains of such intervals $[a, b]$ of a lattice L in which $(\mathbf{B}_2)$ is not satisfied (i. e., in which all maximal chains between a and b are of infinite length). Since for lattices of finite length the property $(\mathbf{A}_2)$ defines the semi-modularity, one would expect that replacing $(\mathbf{A}_2)$ by the general condition of semi-modularity [**2**, p. 204], (**S**) remains valid even without $(\mathbf{B}_2)$. However, this conjecture does not turn out to be right. We prove the following, somewhat surprising theorem:

Theorem 3. *Statement* (**S**) *of Theorem 1 is independent (not only of the semi-modularity but also) of the distributivity of the sublattice $[a, b]$.*

Proof. Since (**S**) obviously does not imply the distributivity of $[a, b]$, it suffices to construct a distributive lattice, naturally of infinite length, in which (**S**) is not satisfied. For this purpose consider the set H of all couples (x_1, x_2) in which x_1 resp. x_2 runs over all real resp. all rational numbers in the closed interval [0, 1], and define a partial ordering in H as follows:

$$(x_1, x_2) \geqq (y_1, y_2) \quad \text{if and only if} \quad x_1 \geqq y_1, \quad x_2 \geqq y_2.$$

Consequently,

$$(x_1, x_2) = (y_1, y_2) \quad \text{if and only if} \quad x_1 = y_1, \quad x_2 = y_2.$$

By this partial ordering H is made into a lattice which obviously satisfies the distributive laws. Let now Θ be the equivalence relation on H defined as follows:

$$(x_1, x_2) \equiv (y_1, y_2) \pmod{\Theta} \quad \text{means} \quad \begin{cases} (x_1, x_2) = (y_1, y_2) \\ \text{or } x_2 = y_2 = 1. \end{cases}$$

Then clearly $(x_1, x_2) \equiv (x_1^*, x_2^*)$, $(y_1, y_2) \equiv (y_1^*, y_2^*) \pmod{\Theta}$ imply $(x_1, x_2) \cap (y_1, y_2) \equiv (x_1^*, x_2^*) \cap (y_1^*, y_2^*) \pmod{\Theta}$ and similarly for $\cup$; that is, Θ is a congruence relation on H. This means, that the set L of all residue classes $\overline{(x_1, x_2)}$ mod Θ forms again a distributive lattice; the greatest element of L is the residue class $\overline{(x_1, 1)}$ $(0 \leqq x_1 \leqq 1)$ and the least element of L is the residue class $\overline{(0, 0)}$. It is now easily shown that (S) does not hold in L. For, the chain

$$\overline{(0, x_2)} \quad (0 \leqq x_2 \leqq 1\,;\ x_2 \text{ rational})$$

is a maximal one between $\overline{(0, 0)}$ and $\overline{(x_1, 1)}$ and it is countable, however the chain consisting of the elements

$$\overline{(x_1, 0)} \quad (0 \leqq x_1 \leqq 1,$$

and

$$\overline{(1, x_2)} \quad (0 \leqq x_2 \leqq 1\,;\ x_2 \text{ rational})$$

is again a maximal one between the same elements of L, but is uncountable. Thus our theorem is proved.

References.

[1] G. Birkhoff, *Lattice theory* (Amer. Math. Soc. Coll. Publ., vol. 25), revised edition (New York, 1948).

[2] R. Croisot, Contribution à l'étude des treillis semi-modulaires de longueur infinie, *Annales Sci. École Norm. Sup.*, **68** (1951), 203—265.

[3] R. Dedekind, Über die von drei Moduln erzeugte Dualgruppe, *Math. Annalen*, **53** (1900), 371—403.

[4] G. Szász, On the structure of semi-modular lattices of infinite length, *these Acta*, **14** (1951/52), 239—245.

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