On the theory of quasi-unitary algebras.

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To Professor L. Kalmár on his 50th birthday.

1. Introduction. J. DIXMIER has recently introduced the concept of the quasi-unitary algebra, and developed various theorems clarifying its structure, cf. [2]. This notion contains as special cases the unitary algebras ([6], [9]), and the group algebras of (not necessarily unimodular) locally compact groups. Also the examples of factors given by J. VON NEUMANN in [8] can be interpreted from this point of view. It can be shown in virtue of this circumstance, that while many properties of the unitary algebras can be extended to the quasi-unitary case, the latter notion is more general also in the respect that the corresponding left ring $\mathbf{R}^g$ need not be semi-finite[1]), i. e. it can possess a nontrivial purely infinite component. This makes a difficulty, in view of the role of the left rings in the investigation of the quasi-unitary algebras. Thus it is of interest to obtain criteria for those quasi-unitary algebras $\mathbf{R}$ for which $\mathbf{R}^g$ is a semi-finite ring, and to clarify their structure.

The present paper is devoted to this problem, and in part continues the investigations of DIXMIER. Firstly, by continuation of his method we prove that if $\mathbf{R}^g$ is semi-finite, then $J=[M'M^{-1}]$[2]), where M is positive, self-adjoint, non-singular, $\eta\mathbf{R}^d$ [3]), and $M'=SMS$ (cf. Theorem 1, for the definitions and notations cf. below **2**). This result combined with DIXMIER's Theorem 2 shows that the representability of J in the form $[M'M^{-1}]$ is a necessary and sufficient condition for the semi-finiteness of $\mathbf{R}^g$, and then $\mathbf{Q}^d \subseteqq \mathbf{P}^d$. In other

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[1]) A ring of operators $\mathbf{N}$ is called semi-finite, if every projection $P \in \mathbf{N}$ contains a finite projection. We say that the projection P is finite if there exists no partial isometry $V \in \mathbf{N}$ with $V^*V=P$, $VV^*=Q<P$.

[2]) Given two (in general unbounded) closed operators S, T defined on a Hilbert space $\mathfrak{H}$, we note by $[ST]$ the minimal closed extension of the product ST (provided it exists).

[3]) If T is a closed operator, we denote by $T\eta\mathbf{N}$ that T commutes with every operator of the commutant $\mathbf{N}'$ of $\mathbf{N}$.

words, Theorem 1 gives a solution for DIXMIER's hypothesis 1 (cf. [2] p. 283) in the semi-finite case.

In Theorem 2 we prove, generalizing the corresponding results for unitary algebras (cf. [6] Theorem 4 and [10] Theorem 18), that if $\mathbf{R}^g$ is semi-finite, then $\mathbf{R}^g$, M' and the maximal extension of the canonical trace can be prescribed. More explicitly, given a semi-finite operator ring $\mathbf{N}$ on a Hilbert space, a positive, self-adjoint, non-singular operator $H\eta\mathbf{N}$, and a maximal normal trace φ defined on a two-sided ideal $\mathfrak{m}\subseteq\mathbf{N}$,[4]) there exists a quasi-unitary algebra $\mathbf{R}$ such that $\mathbf{R}^g$ is $*$-isomorphic with $\mathbf{N}$ and, under this isomorphism, M' and the maximal extension of the canonical trace correspond to H and φ, respectively. If $\mathbf{R}$ is a maximal, then we show that it is determined up to an isomorphism by this choice (Theorem 3).

Next we investigate some properties of the quasi-unitary algebras with a semi-finite $\mathbf{R}^g$. We show beside others that contrary to the unitary case the canonical trace need not be maximal (cf. lemma 10 and the remark which follows). Finally we give a proof for a theorem of DIXMIER about the quasi-central elements, which leads to somewhat more general result (Theorem 5, cf. Theorem 4 in [2]).

2. Definitions and preliminary results. For the following cf. [2], in particular chapters I, II, V—VII. A *quasi-unitary algebra* $\mathbf{R}$ is an algebra over the complex numbers, on which an involutive antiautomorphism $x\to x^s$, an automorphism $x\to x^j$, and an inner product (x, y) are defined, such that $\mathbf{R}$ becomes a pre-Hilbert space satisfying the following axioms:

(i) $(x^s, x^s)=(x, x)$,

(ii) $(x, x^j)\geqq 0$,

(iii) $(xy, z)=(y, x^{js}z)$,

(iv) the mapping $x\to yx$ with fixed y is continuous,

(v) the linear combinations of the elements of the form $xy+(xy)^j$ are dense in $\mathbf{R}$ (x, y, z arbitrary in $\mathbf{R}$).

A *unitary algebra* is a quasi-unitary algebra with $x^j\equiv x$.

Let $\mathfrak{H}_{\mathbf{R}}$ be the Hilbert space, which is obtained by completion of $\mathbf{R}$. By axiom (iv), for every $x\in\mathbf{R}$ there exists a bounded operator U_x (resp. V_x) on $\mathfrak{H}_{\mathbf{R}}$ satisfying $U_x y=xy$ (resp. $V_x y=yx$) for every $y\in\mathbf{R}$. The weak (or

[4]) We recall that a trace φ defined on a two-sided ideal $\mathfrak{m}$ of an operator-ring $\mathbf{N}$ is a positive linear form such that $\varphi(AB)=\varphi(BA)$ for $A\in\mathfrak{m}$ and $B\in\mathbf{N}$. φ is regular, if $\varphi(T)=0$ for $T\in\mathfrak{m}$, $T\geqq 0$ implies $T=0$. A trace φ is normal, if it has the following property: let T_α be an increasing directed set of positive operators $\in\mathfrak{m}$ with a l. u. b. $T\in\mathfrak{m}$, then we have $\varphi(T)=\underset{\alpha}{\text{l. u. b.}}\,\varphi(T_\alpha)$. It is maximal, if it has no proper normal extension. If we say in the following simply a trace, we suppose that it is normal and maximal. In the case of semi-finite rings this means, that it is also regular. Every (not necessarily maximal) trace φ has a maximal extension, which is uniquely determined, if $\mathbf{N}$ is semi-finite and $\mathfrak{m}$ is strongly dense in it. For a theory of traces cf. [1].

strong) closure of the operators U_x (resp. V_x) is a ring of operators $\mathbf{R}^g$ (resp. $\mathbf{R}^d$) on $\mathfrak{H}_\mathbf{R}$ with unit operator. $\mathbf{R}^g$ (resp. $\mathbf{R}^d$) is called the left (right) ring of $\mathbf{R}$. The set of bounded operators on $\mathfrak{H}_\mathbf{R}$ which commute with every element of $\mathbf{R}^g$, coincides with $\mathbf{R}^d$; in other words, $\mathbf{R}^d$ is the commutant of $\mathbf{R}^g$: $\mathbf{R}^{g\prime} = \mathbf{R}^d$ (Theorem of commutation). The minimal closed extension J of the correspondence $x \to x^j$ is positive, self-adjoint, non-singular, and it is equal to the minimal closed extension of its restriction to the linear combinations of the elements xy $(x, y \in \mathbf{R})$. Denoting by S the involution of $\mathfrak{H}_\mathbf{R}$ obtained by the continuation of the correspondence $x \to x^s$ over $\mathfrak{H}_\mathbf{R}$, we have $J^{-1} = SJS$. The mapping $T \to STS$ $(T \in \mathbf{R}^g)$ establishes a conjugate linear isomorphism between $\mathbf{R}^g$ and $\mathbf{R}^d$. An element a of $\mathfrak{H}_\mathbf{R}$ is called left bounded, if there exists an operator U_a defined on $\mathfrak{H}_\mathbf{R}$ such that $U_a x = V_x a$ for every $x \in \mathbf{R}$. Then $U_a \in \mathbf{R}^g$. If a is left bounded and $T \in \mathbf{R}^g$, then Ta is left bounded too, and $U_{Ta} = TU_a$. $U_a^* = U_b$ (with b left bounded) if and only if $a \in D_{J^{-1}}$, and then $b = SJ^{-1}a$.

If $J = [M'M^{-1}]$, where $M \eta \mathbf{R}^d$ is positive, self-adjoint, non-singular, and $M' = SMS$, then $\mathbf{R}^g$ is semi-finite. This follows from the fact, that the elements $A = \sum_{i=1}^{n} U_{a_i} U_{b_i}^*$ $(a_i, b_i$ left bounded and $\in D_M)$ form a strongly dense two-sided ideal $\mathfrak{m} \subseteq \mathbf{R}^g$, and $\varphi(A) = \sum_{i=1}^{n} (Ma_i, Mb_i)$ defines a (not necessarily maximal) trace on $\mathfrak{m}$. This is the *canonical trace* for $\mathbf{R}^g$. The operator M is not determined uniquely by the condition $J = [M'M^{-1}]$. If $C \eta \mathbf{R}^g \cap \mathbf{R}^d$ is positive, self-adjoint and non-singular, then $[CM]$ possesses this property too, and conversely, if $J = [M_1' M_1^{-1}]$, then there exists an operator $C \eta \mathbf{R}^g \cap \mathbf{R}^d$ of the same kind such that $M_1 = [CM]$. Therefore, the canonical trace is also not uniquely determined. Denote by $\mathbf{P}^d$ the set of operators $\in \mathbf{R}^g$ which commute with J, and by $\mathbf{Q}^d$ the set of operators $\in \mathbf{R}^g$ which commute with $\mathbf{P}^d$. Then from $J = [MM^{-1}]$ it follows that $\mathbf{Q}^d \subseteqq \mathbf{P}^d$; and conversely, if $\mathbf{R}^g$ is semi-finite and $\mathbf{Q}^d \subseteqq \mathbf{P}^d$, then $J = [M'M^{-1}]$.

3. Theorem 1. *Let $\mathbf{R}$ be a quasi-unitary algebra for which $\mathbf{R}^g$ is semi-finite. Then $J = [M'M^{-1}]$, where $M \eta \mathbf{R}^d$ is self-adjoint, positive, non-singular, and $M' = SMS$.*

From the proof of Theorem 3 in [2] we shall use the following facts. Let φ be a trace defined on the (strongly dense) two-sided ideal $\mathfrak{m}$ of $\mathbf{R}^g$. Then there exists a positive, self-adjoint, non-singular operator $M \eta \mathbf{R}^d$ such that

a) if we denote by A the set of those left bounded elements $\in \mathfrak{H}_\mathbf{R}$ for which $U_a \in \mathfrak{m}^{\frac{1}{2}}$ [5]), then M is the minimal closed extension of its restriction to A;

[5]) If $\mathfrak{m}$ is a two-sided ideal in an operator-ring $\mathbf{N}$, then $\mathfrak{m}^{\frac{1}{2}}$ denotes the two-sided ideal formed by the elements $T \in \mathbf{N}$ for which $T^*T \in \mathfrak{m}$.

b) if $a, b \in A$ then $(Ma, Mb) = \varphi(U_a U_b^*)$;

c) if J commutes with M, then putting $M' = SMS$ we have $J = [M'M^{-1}]$ (cf. [2] lemma 23).

Therefore in the following it suffices to show that in consequence of *a)* and *b)* J commutes with M.[6])

Before passing to the proof of this statement we need some lemmas.

Lemma 1. *For $x \in \mathbf{R}$ there exists an operator T_x such that $T_x M \subseteq M V_x$.*

Proof: We note first that if $a \in A$ then $V_x a \in A$ too, because $V_x a = U_a x$, and so $U_{V_x a} = U_{U_a x} = U_a U_x \in \mathfrak{m}^{\frac{1}{2}}$. We define $T'_x Ma = M V_x a$ for $a \in A$; T'_x is densely defined. Since $\|T'_x Ma\|^2 = \|M V_x a\|^2 = \varphi(U_a U_x U_x^* U_a^*) \leqq K\varphi(U_a U_a^*) = K\|Ma\|^2$ where K depends only on x, T'_x can be extended by continuity to a bounded operator T_x. Finally, if $a \in D_M$, then there exists a sequence $a_n \in A$ $(n = 1, 2, \ldots)$ such that $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} V_x a_n = V_x a$, $\lim_{n\to\infty} Ma_n = Ma$; hence from $T_x M a_n = M V_x a_n$ $(n = 1, 2, \ldots)$ it follows that $V_x a \in D_M$ and $T_x Ma = M V_x a$.

We introduce for $X, Y \in \mathfrak{m}^{\frac{1}{2}}$ the scalar product $(X, Y) = \varphi(XY^*)$. Then with the involutive antiautomorphism $X \to X^*$ and the usual product $\mathfrak{m}^{\frac{1}{2}}$ becomes a unitary algebra, which we denote again by $\mathfrak{m}^{\frac{1}{2}}$, and its completion by $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$. As it is shown in [6] (Theorem 4) or in different form in [10] (Theorem 18), the left and right rings of $\mathfrak{m}^{\frac{1}{2}}$ can be obtained in the following way. Let $T \in \mathbf{R}^g$ and $X \in \mathfrak{m}^{\frac{1}{2}}$, then we have $\varphi((TX)^*(TX)) = \varphi(X^* T^* TX) \leqq \|T\|^2 \varphi(X^* X)$, hence the correspondence $X \to TX$ can be extended to a bounded linear transformation L_T defined on $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$. The totality of these operators L_T $(T \in \mathbf{R}^g)$ coincides with the left ring of $\mathfrak{m}^{\frac{1}{2}}$. Similarly, we can define for $T \in \mathbf{R}^g$ the operator R_T by extending the correspondence $X \to XT$ to a bounded linear transformation on $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$, and the totality of these operators forms the right ring of $\mathfrak{m}^{\frac{1}{2}}$.

Lemma 2. *The correspondence $Ma \to U_a$ $(a \in A)$ can be extended to an isomorphism ψ between the spaces $\mathfrak{H}_{\mathbf{R}}$ and $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$, which carries $\mathbf{R}^g$ into the left ring, and $\mathbf{R}^d$ into the right ring of $\mathfrak{m}^{\frac{1}{2}}$.*

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[6]) If $H_1 = \int_0^\infty \lambda \, dE_\lambda^{(1)}$ and $H_2 = \int_0^\infty \lambda \, dE_\lambda^{(2)}$ are two operators defined on a Hilbert space $\mathfrak{H}$, we say that they commute, if $E_\lambda^{(1)}$ and $E_\mu^{(2)}$ commute for $\lambda, \mu \geqq 0$. In this case $[H_1 H_2]$ always exists, and is positive, self-adjoint.

Proof: If $X \in \mathfrak{m}^{\frac{1}{2}}$ and $y \in \mathbf{R}$, $U_{Xy} = XU_y \in \mathfrak{m}^{\frac{1}{2}}$, hence $Xy \in A$. Since the set of the operators U_y ($y \in \mathbf{R}$) forms a strongly dense $*$-subalgebra of $\mathbf{R}^g$, there exists, by a theorem of KAPLANSKY (cf. [7] Theorem 1), a directed set of operators $\{U_{y_\alpha}\}_{\alpha \in F}$ ($y_\alpha \in \mathbf{R}$), which converges strongly and boundedly in norm to I. φ is normal, hence the mapping $B \to \varphi(X^* XB)$ ($B \in \mathbf{R}^g$) is continuous on bounded sets of $\mathbf{R}^g$ in the strong topology (cf. [3] Corollary 8 of the Theorem 3). So we have $\lim_\alpha \varphi(X^* X U_{x_\alpha}) = \varphi(X^* X)$. From this it is clear that the linear set of the operators U_a ($a \in A$) is dense in $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$. Since $(Ma, Mb) = \varphi(U_a U_b^*)$ ($a, b \in A$), the correspondence $Ma \to U_a$ can be extended to a unitary mapping ψ between the spaces $\mathfrak{H}_{\mathbf{R}}$ and $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$. If $T \in \mathbf{R}^g$ ($a, b \in A$), then $(TMa, Mb) = (MTa, Mb) = \varphi(U_{Ta} U_b^*) = \varphi(TU_a U_b^*)$. Hence the mapping ψ carries $\mathbf{R}^g$ into the left ring of $\mathfrak{m}^{\frac{1}{2}}$. Since the right ring is the commutant of the left ring (Theorem of commutation), we see that ψ maps at the same time $\mathbf{R}^d$ into the right ring of $\mathfrak{m}^{\frac{1}{2}}$.

We put in the following $C = (M + iI)(M - iI)^{-1}$; C is unitary and $\in \mathbf{R}^d$. To prove that J and M commute, it suffices evidently to show this for J and C.

Lemma 3. *If $a \in A$ then $Ca \in A$ and $U_{Ca} = U_a C'$, where C' is a unitary operator $\in \mathbf{R}^g$, which depends only on C.*

Proof: Since $C \in \mathbf{R}^d$, there exists by lemma 2 a unitary operator $C' \in \mathbf{R}^g$ depending only on C such that $\psi(CMa) = U_a C'$ for $a \in A$. If $U_{x_\alpha} \to C'$ strongly and such that $\|U_{x_\alpha}\| \leqq 1$ (Theorem of KAPLANSKY), then $U_a U_{x_\alpha}$ converges to $U_a C'$ in the metric of the space $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$.[7]) Since $\psi(MV_x a) = U_a U_x$, we have $\lim_\alpha MV_{x_\alpha} a = CMa = MCa$, hence for $y \in \mathbf{R}$ by lemma 1 $\lim_\alpha MV_y V_{x_\alpha} a = \lim_\alpha T_y MV_{x_\alpha} a = T_y MCa = MV_y Ca$. Let $\int_0^\infty \lambda dE_\lambda$ be the spectral representation of M. For every $\delta > 0$ $\|(I - E_\delta) V_y Ca\| = \lim_\alpha \|(I - E_\delta) V_y V_{x_\alpha} a\| \leqq K\|y\|$, where K does not depend on δ and α, because the operators $U_{V_{x_\alpha} a} = U_a U_{x_\alpha}$ are uniformly bounded in norm. Since M is non-singular, we have $\|V_y Ca\| \leqq K\|y\|$ for every $y \in \mathbf{R}$. This proves that Ca is left bounded, and that $U_{Ca} = \psi(MCa) = U_a C'$.

Lemma 4. *For every left bounded $a \in \mathfrak{H}_{\mathbf{R}}$, Ca is left bounded too, and $U_{Ca} = U_a C'$ where C' is a unitary operator $\in \mathbf{R}^g$ which depends only on C.*

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[7]) If $T \in \mathfrak{m}^{\frac{1}{2}}$, $\|U_\alpha\| \leqq 1$, $U_\alpha \to U$ strongly, and U is unitary, then $TU_\alpha \to TU$ in the metric of the space $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$. In this case we have namely

$$\varphi([TU - TU_\alpha][TU - TU_\alpha]^*) =$$
$$= \varphi(TT^*) + \varphi(TU_\alpha U^* T^*) - 2\,\mathrm{Re}\,\varphi(TU_\alpha U^* T^*) \leqq 2(\varphi(TT^*) - \mathrm{Re}\,\varphi(TU_\alpha U^* T^*)) \to 0.$$

Proof: If $T \in \mathfrak{m}^{\frac{1}{2}}$ and a is left bounded, then $Ta \in A$, hence by lemma 3 $U_{CTa} = U_{Ta}C' = TU_aC'$, where $C' \in \mathbf{R}^g$ depends only on C. If $T_\alpha \in \mathfrak{m}^{\frac{1}{2}}$ and $T_\alpha \to I$ strongly, then $V_x Ca = \lim_\alpha V_x T_\alpha Ca = \lim_\alpha T_\alpha U_a C' x = U_a C' x$ for every $x \in \mathbf{R}$, which proves that Ca is left bounded and that $U_{Ca} = U_a C'$.

Similarly, it can be proved that for every left bounded a, $C^* a$ is left bounded too, and that $U_{C^*a} = U_a C'^*$.

Proof of Theorem 1: We have by lemma 4 for every left bounded $a \in D_{J^{-1}}$: $U^*_{Ca} = (U_a C')^* = C'^* U_{SJa^{-1}} = U_{C'^*SJa^{-1}}$, which proves that $Ca \in D_{J^{-1}}$ and $SJ^{-1}Ca = C^* SJ^{-1}a$. Substituting a by Sx $(x \in \mathbf{R})$, and putting $\overline{C} = SCS$, $C'' = C'^*$, we get $J\overline{C}x = C''Jx$ for every $x \in \mathbf{R}$. If a is arbitrary $\in D_J$, then we can determine a sequence $x_n \in \mathbf{R}$ such that $x_n \to a$ and $Jx_n \to Ja$, since J is the minimal closed extension of its restriction to $\mathbf{R}$. Therefore $\lim_{n\to\infty} J\overline{C}x_n = \lim_{n\to\infty} C''Jx_n = C''Ja$, which proves that if $a \in D_J$, then $\overline{C}a \in D_J$ too. Replacing C by C^* in the above reasoning, and noting that $\overline{C^*} = \overline{C}^*$, we get that $a \in D_J$ gives $\overline{C}^* a \in D_J$ too. Therefore the domains of definition of the operators J and $\overline{C}^* J \overline{C}$ coincide, and since C'' is unitary, we have $\|\overline{C}^* J \overline{C} a\|^2 = \|Ja\|^2$ for $a \in D_J$. But as it is known, this gives necessarily $J = \overline{C}^* J \overline{C}$ or $J^{-1} = C^* J^{-1} C$, and so the proof of Theorem 1 is completed.

Corollary. *If* $\mathbf{R}$ *is a quasi-unitary algebra with a semi-finite* $\mathbf{R}^g$, *then* $\mathbf{Q}^d \subseteq \mathbf{P}^d$.

Proof: This is an immediate consequence of our Theorem 1 and of Theorem 2 in [2].

4. Theorem 2. *Given a semi-finite ring of operators* $\mathbf{N}$ *defined on a Hilbert space* $\mathfrak{H}$, *a positive, self-adjoint, non-singular operator* $H \eta \mathbf{N}$, *a trace* φ *defined on a two-sided ideal* $\mathfrak{m}$ *of* $\mathbf{N}$, *there exists a quasi-unitary algebra* $\mathbf{R}$ *with the following properties:* $\mathbf{R}^g$ *is* $*$*-isomorphic with* $\mathbf{N}$; M' (cf. Theorem 1) *and the maximal extension of the canonical trace on* $\mathbf{R}^g$, *and* H, φ, *respectively, correspond to each other under this isomorphism.*

Proof: Let $\int_0^\infty \lambda dE_\lambda$ $(E_\lambda \in \mathbf{N})$ be the spectral representation of H. We denote by $\mathbf{R} \subseteq \mathfrak{m}^{\frac{1}{2}}$ the $*$-subalgebra of $\mathbf{N}$ consisting of the operators $X \in \mathfrak{m}^{\frac{1}{2}}$, for which there exists a projection $E(\Delta) = E_{\lambda_2} - E_{\lambda_1}$, $\Delta = (\lambda_1, \lambda_2)$, $0 < \lambda_1 < \lambda_2$, with $X = E(\Delta)X = XE(\Delta)$. For such an interval Δ we say that it contains the operator X, and we denote by $\mathbf{R}_\Delta$ the totality of operators $\in \mathbf{R}$ contained by Δ. It is evident that for $X, Y \in \mathbf{R}$ there exists a Δ such that $X, Y \in \mathbf{R}_\Delta$. We define an inner product between X, Y by $(X, Y) = \varphi\big((XH^{-1}(\Delta))(XH^{-1}(\Delta))^*\big)$, where $H^{-1}(\varDelta)=\int\limits_{\varDelta}\lambda^{-1}dE_\lambda$.[8]) This is obviously independent of $\varDelta$, provided that it contains X and Y. It is not hard to verify that with this definition $\mathbf{R}$ becomes a pre-Hilbert space. To make a quasi-unitary algebra from $\mathbf{R}$ we define an automorphism of $\mathbf{R}$ by $X^j=H(\varDelta)XH^{-1}(\varDelta)$, and an involutive antiautomorphism by $X^s=H^{-1}(\varDelta)X^*H(\varDelta)$, where $X\in\mathbf{R}_\varDelta$. To show that the operation ^{j} gives an automorphism of $\mathbf{R}$, we have to prove that $(aX+BY)^j=$ $=aX^j+BY^j$ for arbitrary complex numbers a, b, and that $(XY)^j=X^jY^j$. Suppose that $X,Y\in\mathbf{R}_\varDelta$, then $(aX+BY)^j=H(\varDelta)(aX+BY)H^{-1}(\varDelta)=$ $=aH(\varDelta)XH^{-1}(\varDelta)+BH(\varDelta)YH^{-1}(\varDelta)=aX^j+BY^j$ by the definition of X^j and Y^j. If $X,Y\in\mathbf{R}_\varDelta$, then $(XY)^j=H(\varDelta)XYH^{-1}(\varDelta)=H(\varDelta)XE(\varDelta)YH^{-1}(\varDelta)=$ $=H(\varDelta)XH^{-1}(\varDelta)H(\varDelta)YH^{-1}(\varDelta)=X^jY^j$. The proof that $X\to X^s$ defines an involutive antiautomorphism of $\mathbf{R}$ is quite similar, and we omit it.

Lemma 5. *With the above definitions* $\mathbf{R}$ *satisfies axioms* (i)—(iv) *of a quasi-unitary algebra, enumerated in* **2**.

Proof: Ad (i): If $X,Y\in\mathbf{R}_\varDelta$, then we have $X^*,Y^*\in\mathbf{R}_\varDelta$ too, and

$$(Y^s,X^s)=\varphi([H^{-1}(\varDelta)Y^*H(\varDelta)H^{-1}(\varDelta)][H^{-1}(\varDelta)X^*H(\varDelta)H^{-1}(\varDelta)]^*)=$$
$$=\varphi(H^{-1}(\varDelta)Y^*XH^{-1}(\varDelta))=\varphi([XH^{-1}(\varDelta)][YH^{-1}(\varDelta)]^*)=(X,Y).$$

Ad (ii): If $X\in\mathbf{R}_\varDelta$ then

$$(X^j,X)=\varphi([H(\varDelta)XH^{-1}(\varDelta)H^{-1}(\varDelta)][XH^{-1}(\varDelta)]^*)=$$
$$=\varphi(H^{\frac{1}{2}}(\varDelta)XH^{-3}(\varDelta)X^*H^{\frac{1}{2}}(\varDelta))\geqq 0.$$ [9])

Ad (iii):

$$(XY,Z)=\varphi([XYH^{-1}(\varDelta)][ZH^{-1}(\varDelta)]^*)=$$
$$=\varphi([YH^{-1}(\varDelta)][X^*ZH^{-1}(\varDelta)])=(Y,X^{js}Z),$$

provided that $X,Y,Z\in\mathbf{R}_\varDelta$, because

$$X^{js}=H(\varDelta)[H^{-1}(\varDelta)X^*H(\varDelta)]H^{-1}(\varDelta)=X^*.$$

Ad (iv):

$$\|XY\|^2=\varphi(XYH^{-2}(\varDelta)Y^*X^*)=$$
$$=\varphi([YH^{-1}(\varDelta)]^*X^*X[YH^{-1}(\varDelta)])\leqq K\varphi(YH^{-1}(\varDelta)][YH^{-1}(\varDelta)]^*)=K\|Y\|^2,$$

where K depends only on X, and $X,Y\in\mathbf{R}_\varDelta$.

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8) In the following we put $H^k(\varDelta)=\int\limits_{\varDelta}\lambda^k dE_\lambda$, $k\gtrless 0$, for an interval $\varDelta=(\lambda_1,\lambda_2)$, $0<\lambda_1<\lambda_2$ and for $H=\int\limits_0^\infty\lambda dE_\lambda$.

9) Note that if $A,B\in\mathfrak{m}^{\frac{1}{2}}$, then $AB\in\mathfrak{m}$ and $\varphi(AB)=\varphi(BA)$. Cf. for example [6], lemme 12.

The proof, that $\mathbf{R}$ satisfies also (v), will be given later (cf. lemma 7).

We take now the completion of $\mathbf{R}$ and denote it by $\mathfrak{H}_{\mathbf{R}}$. Axiom (iv) allows to form the left multiplication operator U_X for every $X \in \mathbf{R}$. By axiom (iii) $U_X^* = U_{X^{js}}$, which shows, that the totality of these operators is $*$-invariant. We denote its weak closure by $\mathbf{R}^g$.

Lemma 6. *$\mathbf{R}^g$ contains the unit operator and is $*$-isomorphic with $\mathbf{N}$.*

Proof: As in lemma 2 we use again the unitary algebra formed by aid of $\mathfrak{m}^{\frac{1}{2}}$, and denote its completion by $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$. As we mentioned it, its left ring $\left(\mathfrak{m}^{\frac{1}{2}}\right)^g$ coincides with the totality of the operators of the form L_T, $T \in \mathbf{N}$, and the correspondence $T \rightarrow L_T$ establishes obviously a $*$-isomorphism between $\mathbf{N}$ and $\left(\mathfrak{m}^{\frac{1}{2}}\right)^g$. Hence it suffices to show that there exists a unitary mapping ψ between the spaces $\mathfrak{H}_{\mathbf{R}}$ and $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$ which carries $\mathbf{R}^g$ into $\left(\mathfrak{m}^{\frac{1}{2}}\right)^g$. We define now $\psi'(X) = XH^{-1}(\Lambda)$ for $X \in \mathbf{R}$ and Λ containing X. We have evidently $(X, Y) = (\psi'(X), \psi'(Y))_1$. (We denote by $(\,,)_1$ the inner product in $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$ to avoid the confusion with the inner product in $\mathfrak{H}_{\mathbf{R}}$.) Since $\mathbf{R}$ as a linear set is dense in $\mathfrak{H}_{\mathbf{R}}$ by definition, and, as it is easily seen, so is in the space $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$, and since $\psi'(\mathbf{R}) = \mathbf{R}$, ψ' can be extended to a unitary mapping ψ between these spaces. If $T, X, Y \in \mathbf{R}$,

$$(U_T X, Y) = (TX, Y) = (\psi(TX), \psi(Y))_1 = (L_T \psi(X), \psi(Y))_1$$

and since $\mathbf{R}$ as a $*$-subalgebra is dense in $\mathbf{R}^g$ and $\left(\mathfrak{m}^{\frac{1}{2}}\right)^g$, lemma 6 is proved.[10])

We denote by J the minimal closed extension of the correspondence $X \rightarrow X^j$ in $\mathfrak{H}_{\mathbf{R}}$, and by S the involution obtained by the extension of $X \rightarrow X^s$.

Lemma 7. *$J = [M'M^{-1}]$, where $M' \eta \mathbf{R}^g$ corresponds to H under the $*$-isomorphism between $\mathbf{R}^g$ and $\mathbf{N}$, and $M = SM'S$.*

Proof: We denote by $\bar{H}'$ the operator in $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$, corresponding to H under the $*$-isomorphism between $\left(\mathfrak{m}^{\frac{1}{2}}\right)^g$ and $\mathbf{N}$. Let $\bar{S}$ be the involution obtained by the continuation of $X \rightarrow X^*$ over $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$, and $\bar{H} = \bar{S}\bar{H}'\bar{S}$. $\bar{H}'$ and $\bar{H}$ are commuting, selfadjoint, non-singular operators, hence $[\bar{H}'\bar{H}^{-1}]$ exists. The isomorphism ψ carries the correspondence $X \rightarrow X^j$ $(X \in \mathbf{R})$ in $\mathfrak{H}_{\mathbf{R}}$ into the linear transformation defined on $\mathbf{R} \subseteq \mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$ by $J'X = H(\Lambda)XH^{-1}(\Lambda)$ $(X \in \mathbf{R}_\Lambda)$. It is clear that $J'X = \bar{H}'\bar{H}^{-1}X$.

We prove now, that the minimal closed extension of J' is identical with $[\bar{H}'\bar{H}^{-1}]$, or, that the latter operator is the minimal closed extension of its

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[10]) Observe that because of the maximality of φ, $\mathfrak{m}^{\frac{1}{2}}$ is strongly dense in $\mathbf{N}$.

restriction to $\mathbf{R} \subseteq \mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$. But this is contained in the following statement: Let $X \in \mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$ be in the domain of definition of $\bar{H}'\bar{H}^{-1}$, then there exists an element $X_1 \in \mathbf{R}$ such that X_1 and $\bar{H}'\bar{H}^{-1}X_1$ are arbitrarily near to X and $\bar{H}'\bar{H}^{-1}X$, respectively. To see this, we put $\bar{H}' = \int_0^\infty \lambda d\bar{E}'_\lambda \left(\bar{E}'_\lambda \in \left(\mathfrak{m}^{\frac{1}{2}}\right)''\right)$, and $\bar{E}_\lambda = \bar{S}\bar{E}'_\lambda\bar{S}$. Then there exists two intervals, Δ_1 and Δ_2, of the form $0 < \lambda_1 \leqq \lambda \leqq \lambda_2$ such that $\bar{E}'(\Delta_1)\bar{E}(\Delta_2)X$ and $\bar{H}'\bar{H}^{-1}\bar{E}'(\Delta_1)\bar{E}(\Delta_2)X$ are arbitrary near to X and $\bar{H}'\bar{H}^{-1}X$ respectively, and an element $X_1 \in \mathbf{R}$, such that X_1 and $\bar{H}'\bar{H}^{-1}X_1$ are arbitrary near to $\bar{E}'(\Delta_1)\bar{E}(\Delta_2)X$ and $\bar{H}'\bar{H}^{-1}\bar{E}'(\Delta_1)\bar{E}(\Delta_2)X$, respectively. If M' and M correspond in the space $\mathfrak{H}_{\mathbf{R}}$ to $\bar{H}'$ and $\bar{H}$ respectively, then we have obviously $J = [M'M^{-1}]$, $M'\eta\mathbf{R}''$, and even $M = SM'S$. For this it suffices to observe that for $X \in \mathbf{R}$ and a suitable interval Δ,

$$\psi(SX) = (H^{-1}(\Delta)X^*H(\Delta))H^{-1}(\Delta) = (XH^{-1}(\Delta))^* = \bar{S}\psi(X),$$

i. e. $\bar{S}$ and S correspond to each other under ψ.

So the proof of lemma 7 is completed.

To show that $\mathbf{R}$ satisfies also the axiom (v) of the quasi-unitary algebras (cf. **2**), we need the folloving

Lemma 8. *J is the minimal closed extension of its restriction to the linear set in $\mathfrak{H}_{\mathbf{R}}$ formed by the linear combination of the elements of the form XY $(XY \in \mathbf{R})$.*

Proof: Passing by aid of the unitary mapping ψ to a problem in $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$, it suffices to prove the following assertion: Given $X \in \mathbf{R}_\Delta$, there exists a projection $P \leqq E(\Delta)$, $P \in \mathfrak{m}^{\frac{1}{2}}$, such that $PX \in \mathbf{R}$ is arbitrary near to X in the metric of $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$. In this case $[\bar{H}'\bar{H}^{-1}]PX = H(\Delta)PXH^{-1}(\Delta)$ is arbitrary near to $[\bar{H}'\bar{H}^{-1}]X = H(\Delta)XH^{-1}(\Delta)$, and it was shown in the preceding lemma, that $[\bar{H}'\bar{H}^{-1}]$ is the minimal closed extension of its restriction to $\mathbf{R}$. But this follows from the fact, that $E(\Delta)$ is the l. u. b. of projections in $\mathfrak{m}^{\frac{1}{2}}$[10]), and from the remark in the footnote[7]).

An immediate consequence of this lemma is that $\mathbf{R}$ satisfies also axiom (v) in (**2**). By lemma 7 namely J is the minimal closed extension of the product of two commuting, positive, self-adjoint operators M^{-1} and M', hence it is itself positive, self-adjoint. So the range of $I+J$ is $\mathfrak{H}_{\mathbf{R}}$, which combined with lemma 8 gives plainly that the elements $(I+J)XY = (XY) + (XY)^J$ $(XY \in \mathbf{R})$ span $\mathfrak{H}_{\mathbf{R}}$.

Summing up the above results, it can be seen from lemma 5 and from the preceding remark, that with the definitions at the beginning of the present section $\mathbf{R}$ becomes a quasi-unitary algebra. Lemma 6 shows that $\mathbf{R}''$ is $*$-isomorphic with $\mathbf{N}$; by lemma 7 we have $J = [M'M^{-1}]$ with $M\eta\mathbf{R}''$, $M = SM'S$, and M' corresponds to H under this isomorphism. To conclude the proof of the Theorem 2 it remains only to show, that the maximal extension φ' of the canonical trace in $\mathbf{R}^g$ corresponds to φ. If $X, Y \in \mathbf{R}_A$, then one sees at once, that $X, Y \in D_M$, and $\varphi'(U_X U_Y^*) = (MX, MY) = (XH(A), YH(A)) = \varphi(XY^*)$. It suffices therefore to remark that the trace φ is uniquely determined by its values on elements of the form XY^* $(X, Y \in \mathbf{R})$. But if the projections $P_\alpha \in \mathbf{R}$ converges to I strongly, and $X \in \mathfrak{m} \subset \mathfrak{m}^{\frac{1}{2}} \subset \mathbf{N}$, then

$$\varphi(X) = \lim_\alpha \varphi(P_\alpha X) = \lim_\alpha \varphi(P_\alpha X P_\alpha P_\alpha),$$

and $P_\alpha X P_\alpha \in \mathbf{R}$, qu. e. d.

5. In the following we consider a quasi-unitary algebra $\mathbf{R}$ with a semi-finite left ring and we denote by φ the maximal extension of the canonical trace, and by $\mathfrak{m} \subset \mathbf{R}^g$ the corresponding two-sided ideal. We recall that now $J = [M'M^{-1}]$ (cf. Theorem 1) and we put $M = \int_0^\infty \lambda \, dE_\lambda$ $(E_\lambda \in \mathbf{R}^d)$ and $M' = \int_0^\infty \lambda \, dE'_\lambda$, $E'_\lambda = SE_\lambda S \in \mathbf{R}^g$.

Lemma 9. *Suppose* $T \in \mathfrak{m}^{\frac{1}{2}}$ *be such that* $TE'(A) = T$, *where* $E'(A) = E'_{\lambda_2} - E'_{\lambda_1}$, $0 < \lambda_1 < \lambda_2$. *Then* $T = U_a$, *where* a *ist left bounded.*

Proof: By the theorem of KAPLANSKY (cf. [7] Theorem 1) we can choose a directed set of elements $x_\alpha \in \mathbf{R}$ such that $\| U_{x_\alpha} \| \leqq 1$, and

$$\text{weak} \lim_\alpha U_{x_\alpha} = I.$$

We need now the following result of DIXMIER (cf. [2] lemme 7.a): If $a \in D_{J^{-1}}$ is left bounded and $L \in \mathbf{R}^g$ commutes with J, then SL^*Sa is left bounded too, and $U_{SL^*Sa} = U_a L$. We put $TE(A)x_\alpha = a_\alpha$, and prove that these elements converge weakly to an element $a \in \mathfrak{H}_\mathbf{R}$. Applying the above lemma to the case $a = x_\alpha$, $L = M^{-1}(A)$, we see that $M^{-1}(A)x_\alpha$ is left bounded, and $U_{M^{-1}(A)x_\alpha} = U_{x_\alpha} M'^{-1}(A)$. Hence it follows that $M^{-1}(A)a_\alpha = TM^{-1}(A)x_\alpha$ is left bounded too, and $U_{M^{-1}(A)a_\alpha} = TU_{M^{-1}(A)x_\alpha} = TU_{x_\alpha} M'^{-1}(A)$. So we have

$$\| a_\alpha \|^2 = \| MM^{-1}(A) a_\alpha \|^2 = \varphi(U_{M^{-1}(A)a_\alpha} U^*_{M^{-1}(A)a_\alpha}) =$$
$$= \varphi(TU_{x_\alpha} M'^{-2}(A) U^*_{x_\alpha} T^*) \leqq \| U_{x_\alpha} \|^2 \| M'^{-2}(A) \| \varphi(TT^*) \leqq K.$$

where K does not depend on α. We have further $\lim_\alpha (a_\alpha, xy) = \lim_\alpha (U_{a_\alpha} y^{sj}, x) = (Ty^{sj}, x)$ for $x, y \in \mathbf{R}$, which proves our assertion since the linear combinations of the elements xy $(x, y \in \mathbf{R})$ are dense in $\mathfrak{H}_\mathbf{R}$. Finally

$$Tx = \text{weak} \lim_\alpha U_{a_\alpha} x = \text{weak} \lim_\alpha V_x a_\alpha = V_x a$$

for $x \in \mathbf{R}$, from which lemma 9 follows immediately.

We recall ([2]), VIII) that the quasi-unitary algebra $\mathbf{R}$ is a continuation of $\mathbf{R}'$, if $\mathbf{R}'$ is a subalgebra of $\mathbf{R}$, the inner product, the automorphism and the involutive antiautomorphism in $\mathbf{R}'$ are the restrictions of the corresponding notions in $\mathbf{R}$, and finally $\mathbf{R}'$ is dense in $\mathbf{R}$. $\mathbf{R}$ is said to be maximal,[11]) if it has no proper continuation. $\mathbf{R}$ is maximal if and only if it possesses the following property: If for $a \in \mathfrak{H}_{\mathbf{R}}$ $J^n a$ exists and is left bounded for every $n \gtrless 0$, then $a \in \mathbf{R}$. Every quasi-unitary algebra is contained in a (uniquely determined) maximal algebra.

Theorem 3. *Let $\mathbf{R}_1$ and $\mathbf{R}_2$ be maximal quasi-unitary algebras, with semi-finite left rings $\mathbf{R}_1^g$ and $\mathbf{R}_2^g$. We put $J_1 = [M_1' M_1^{-1}]$ and $J_2 = [M_2' M_2^{-1}]$* (cf. Theorem 1). *Suppose there exists a $*$-isomorphism ω between $\mathbf{R}_1^g$ and $\mathbf{R}_2^g$ such that M_1' and M_2', the maximal extensions of the canonical traces φ_1 and φ_2 correspond to each other respectively. Then $\mathbf{R}_1$ and $\mathbf{R}_2$ are isomorphic, i. e. there exists a one to one mapping between them, which preserves the algebraic operations, the inner product, the automorphism, and the involutive antiautomorphism.*

Proof: Denote by $\mathfrak{m}_1$ the two-sided ideal belonging to φ_1, which is strongly dense in $\mathbf{R}_1^g$. We put $M_1 = \int_0^\infty \lambda \, dE_\lambda^{(1)}$ and $E_1(\varDelta) = E_{\lambda_2}^{(1)} - E_{\lambda_1}^{(1)}$, $E_1'(\varDelta) = S_1 E_1(\varDelta) S_1$ for an interval $\varDelta = (\lambda_1, \lambda_2)$, $0 < \lambda_1 < \lambda_2$. We denote further by $\mathbf{S}_1$ the totality of operators $T \in \mathfrak{m}_1^{\frac{1}{2}}$ for which there exists a $\varDelta$ such that $E_1'(\varDelta) T = T E_1'(\varDelta) = T$.

Similarly, we denote by $\mathfrak{m}_2$ the two-sided ideal $\subset \mathbf{R}_2^g$ belonging to φ_2, and we put $M_2 = \int_0^\infty \lambda \, dE_\lambda^{(2)}$, $E_2(\varDelta) = E_{\lambda_2}^{(2)} - E_{\lambda_1}^{(2)}$, $E_2'(\varDelta) = S_2 E_2(\varDelta) S_2$. Let $\mathbf{S}_2$ be the totality of operators $T \in \mathfrak{m}_2^{\frac{1}{2}}$, for which $E_2(\varDelta) T = T E_2(\varDelta) = T$ with a suitable $\varDelta$.

If $T \in \mathbf{S}_1$, then by lemma 9 $T = U_a$, where a is left bounded. The reasoning of this lemma gives that $E_1'(\varDelta) a = E_1(\varDelta) a = a$ for a suitable $\varDelta$. Since $J_1 = [M_1' M_1^{-1}]$, we have $a \in D_{J_1^n}$ for $n \gtrless 0$, and $J_1^n a = M_1'^n(\varDelta) M_1^{-n}(\varDelta) a$. This shows that the totality of these elements a is a subset of $\mathbf{R}_1$, which we denote by $\mathbf{R}_1'$. We have further for $A = U_a$, $B = U_b \in \mathbf{S}_1$:

$$AB = U_{ab} \in \mathbf{S}_1 \quad \text{and} \quad \alpha A + \beta B = U_{\alpha a + \beta b} \in \mathbf{S}_1$$

for arbitrary complex numbers α and β. This gives that $\mathbf{R}_1'$ is a subalgebra of $\mathbf{R}_1$. Observing that $\mathbf{S}_1$ is a $*$-subalgebra of $\mathbf{R}_1^g$, for $a \in \mathbf{R}_1'$ we have $a^{J*} \in \mathbf{R}_1'$, and since plainly $a^{J^n} \in \mathbf{R}_1'$ $(n \lesseqgtr 0)$, we have $a^s \in \mathbf{R}_1'$ too. We form also in a similar way the corresponding $\mathbf{R}_2' \subset \mathbf{R}_2$. If $T \in \mathbf{S}_1$ then $\omega(T) \in \mathbf{S}_2$, because from $T \in \mathfrak{m}_1^{\frac{1}{2}}$ it follows that $\omega(T) \in \mathfrak{m}_2^{\frac{1}{2}}$, and since $\omega(E_1'(\varDelta)) = E_2'(\varDelta)$, $E_1'(\varDelta) T = T E_1'(\varDelta) = T$ gives $E_2'(\varDelta) \omega(T) = \omega(T) E_2'(\varDelta) = \omega(T)$. Define

[11]) In the terminology of Dixmier „algèbre quasi-unitaire achevée".

now a correspondence ψ between $\mathbf{R}_1'$ and $\mathbf{R}_2'$ by $U_{\psi(a)} = \omega(U_a)$, $a \in \mathbf{R}_1'$. Since $\omega(\mathbf{S}_1) = \mathbf{S}_2$ we have $\psi(\mathbf{R}_1') = \mathbf{R}_2'$ too, and it is clear that ψ is one to one. We have plainly $\psi(\alpha a + \beta b) = \alpha\psi(a) + \beta\psi(b)$ and $\psi(ab) = \psi(a)\psi(b)$. To prove that ψ defines an isomorphism between $\mathbf{R}_1'$ and $\mathbf{R}_2'$ it remains to show that (i) $\psi(a^j) = \psi^j(a)$, (ii) $\psi(a^s) = \psi^s(a)$, (iii) $(a, b) = (\psi(a), \psi(b))$.

Ad (i): For $a \in \mathbf{R}_1'$ we have, applying again the lemma 7a in [2]:

$$\omega(U_{a^j}) = \omega(U_{M_1'(A) M_1'^{-1}(A) a}) =$$
$$= \omega(M_1'(A) U_a M_1'^{-1}(A)) = M_2'(A) U_{\psi(a)} M_2'^{-1}(A) = U_{\psi^j(a)}, \text{ hence } \psi(a^j) = \psi^j(a).$$

Ad (ii): $\omega(U_{a^{js}}) = \omega(U_a^*) = \omega(U_a)^* = U_{\psi(a)}^* = U_{\psi^{js}(a)}$, hence $\psi(a^{js}) = \psi^{js}(a)$, which combined with (i) shows that $\omega(a^s) = \omega^s(a)$.

Ad (iii): Since φ_1 and φ_2 correspond to each other under ω we have for $a, b \in \mathbf{R}_1'$ and a suitable A:

$$(a, b) = (M_1 M_1^{-1}(A) a,\ M_1 M_1^{-1}(A) b) = \varphi_1(U_{M^{-1}(A)a} U^*_{M^{-1}(A)b}) =$$
$$= \varphi_1(U_a M'^{-1}(A)\,[U_b M'^{-1}(A)]^*) = \varphi_2(U_{\psi(a)} M_2'^{-1}(A)\,[U_{\psi(b)} M_2'^{-1}(A)]^*) =$$
$$= \varphi_2(U_{M'_2{}^{-1}(A)\psi(a)} U^*_{M'_2{}^{-1}(A)\psi(b)}) = (M_2 M_2^{-1}(A)\psi(a),\ M_2 M_2^{-1}(A)\psi(b)) = (\psi(a), \psi(b)).$$

It is evident that $\mathbf{R}_1'$ (resp. $\mathbf{R}_2'$) is dense in $\mathbf{R}_1$ (resp. $\mathbf{R}_2$). Since a quasi-unitary algebra determines uniquely the corresponding maximal algebra, the above isomorphism can be extended to an isomorphism between $\mathbf{R}_1$ and $\mathbf{R}_2$, qu. e. d.

6. In this section we intend to give more precise information about the structure of the space $\mathfrak{H}_{\mathfrak{m}^{\frac{1}{2}}}$ introduced in **3**, and its connection with the quasi-unitary algebra $\mathbf{R}$. The following lemma is in close connection with the extended Riesz—Fischer theorem (cf. [10] Theorem 13, for the techniques used cf. [4], especially 5 and 6).

Lemma 10. *Let $\mathbf{N}$ be a semi-finite operator-ring on the Hilbert space $\mathfrak{H}$ and φ a trace defined on the two-sided ideal $\mathfrak{m} \subset \mathbf{N}$. Let $T \eta \mathbf{N}$ be a closed operator on $\mathfrak{H}$ and let $T = VH$ be its polar representation, where $H = \int_0^\infty \lambda dE_\lambda$. Putting $H_n = \int_0^n \lambda dE_\lambda$ $(n = 1, 2, \dots)$ and $T_n = VH_n$ we consider the totality of operators $T \eta \mathbf{N}$, for which $T_n \in \mathfrak{m}^{\frac{1}{2}}$ and $\lim_{n\to\infty} \varphi(T_n^* T_n) = \|T\|_2^2$ exists, and denote it by $Q(\mathbf{N})$. Defining the addition for $T, S \in Q(\mathbf{N})$ as $[T+S]$, $Q(\mathbf{N})$ becomes a linear space, and with the scalar product $(S, T) = \lim_{n\to\infty} \varphi(S_n T_n^*)$ even a Hilbert space.*

Proof: By a result of R. Pallu de la Barrière (cf. his Thesis, cited in [2] p. 293) there exists a family of elements $\{a_\alpha\}_{\alpha \in F}$ in $\mathfrak{H}$, such that for

$B \in \mathfrak{m}$ we have $\varphi(B) = \Sigma_{\alpha\in F}(Ba_\alpha, a_\alpha)$. If $T \in Q(\mathbf{N})$, then for $n = 1, 2, \ldots$

$$\varphi(T_n^* T_n) = \Sigma_{\alpha\in F} \| T_n a_\alpha \|^2 \leqq \lim_{n\to\infty} \varphi(T_n^* T_n) = \| T \|_2^2 < +\infty,$$

from which it follows at once that $a_\alpha \in D_T$ $(\alpha \in F)$ and $\| T \|_2^2 = \Sigma_{\alpha\in F} \| Ta_\alpha \|^2$. Obviously the converse is also true: If for $T \eta \mathbf{N}$, $a_\alpha \in D_T$ and $\Sigma_{\alpha\in F} \| Ta_\alpha \|^2 < +\infty$, then $\lim_{n\to\infty} \varphi(T_n^* T_n)$ exists, hence $T \in Q(\mathbf{N})$. If $B = V|B|$ is the polar representation of $B \in \mathfrak{m}^{\frac{1}{2}}$, we have

$$\varphi(BB^*) = \varphi(V|B|^2 V^*) = \varphi(V^* V|B|^2) = \varphi(|B|^2) = \varphi(B^* B),$$

from which it is easily deduced, that if $T \in Q(\mathbf{N})$ then $T^* \in Q(\mathbf{N})$ too and $\| T \|_2^2 = \| T^* \|_2^2$. To prove that $[T+S]$ exists we observe that the linear set A of the elements $\Sigma_{\alpha\in F} B'_\alpha a_\alpha$, where $B'_\alpha \in \mathbf{N}'$ and $B_\alpha \neq 0$ only for a finite set of values of α, is dense in $\mathfrak{H}$. Otherwise because of the regularity of φ there would exist a projection $P \neq 0$, $P \in \mathfrak{m}^{\frac{1}{2}}$, such that $Pa_\alpha = 0$ $(\alpha \in F)$. Then $\varphi(P) = 0$, and so $P = 0$, which yields a contradiction. Since $a_\alpha \in D_{T+S}$ and $a_\alpha \subseteqq D_{T^*+S^*} \subseteqq D_{(T+S)^*}$, therefore $T+S$ and $(T+S)^*$ are densely defined in $\mathfrak{H}$. So $[T+S]$ exists, and plainly $\in Q(\mathbf{N})$. To prove the validity of the associative law $[R+[S+T]] = [[R+S]+T]$, it suffices to show that $T \in Q(\mathbf{N})$ is equal to the minimal closed extension T_1 of its restriction to A. For this observe first that $T_1 \in Q(\mathbf{N})$ too, and we can suppose T to be positive hermitian. If $B \in \mathbf{N}$ and $T \in Q(\mathbf{N})$ then $\lim_{n\to\infty} \varphi(T_n B B^* T_n^*)$ exists. Hence $\lim_{n\to\infty} \varphi(B^* T_n^* T_n B)$ exists too, which implies that $Ba_\alpha \in D_T$, or that TB is densely defined. Suppose now that the range of $I + T_1$ is not dense in $\mathfrak{H}$. Then there exists necessarily a projection $P \neq 0$, $P \in \mathbf{N}$, such that $O \supseteqq P(I+T_1)$. We have

$$O = [P(I+T_1)]^* \supseteqq (I+T_1)^* P \supseteqq (I+T_1^*)P \supseteqq (1+T)P.$$

By the former remark there exists an element $f \neq 0$, $f \in \mathfrak{H}$, such that $Pf = f$ and $f \in D_T$. Hence $(I+T)f = 0$, which is impossible if T is positive hermitian. So the range of $I + T_1$ is dense in $\mathfrak{H}$, which proves that T_1 is hermitian, and so $T = T_1$.

It is easily verified that $(S, T) = \lim_{n\to\infty} \varphi(S_n T_n^*)$ possesses the properties of an inner product; to prove lemma 10 we have therefore only to establish the completeness of this space with respect to the norm $\| \ \|_2$. Let L_n be a sequence $\in Q(\mathbf{N})$ such that

$$\lim_{m,\,n\to\infty} \| L_n - L_m \|_2^2 = \lim_{m,\,n\to\infty} \| L_n^* - L_m^* \|_2^2 = 0.$$

It is easily seen that by $Sf = \lim L_n f$ $(f \in A)$ a (densely defined) linear transformation S is given. We define similarly $S'f = \lim L_n^* f$, $f \in A$. For $f, g \in A$ we have

$$(Sf, g) = \lim_{n\to\infty} (L_n f, g) = \lim_{n\to\infty} (f, L_n^* g) = (f, S'g)$$

This gives that $S^* \supseteq S'$. Thus the minimal closed extension $\bar{S}$ of S exists, and plainly $\bar{S}\eta\mathbf{N}$. In particular $\bar{S}a_\alpha = \lim_{n\to\infty} T_n a_\alpha$ ($\alpha \in F$), from which it follows that $\Sigma_{\alpha\in F}\|\bar{S}a_\alpha\|^2 < +\infty$ and so $\bar{S}\in Q(\mathbf{N})$. Finally it is clear that $\lim_{n\to\infty}\|T_n - \bar{S}\|_2^2 = = \lim_{n\to\infty}\Sigma_{\alpha\in F}\|(T_n - \bar{S})a_\alpha\|^2 = 0$. So the proof of lemma 10 is completed.

In the following we call the elements $\in Q(\mathbf{N})$ square integrable with respect to φ.

Lemma 11. *Let $\mathbf{R}$ be a quasi-unitary algebra with a semi-finite $\mathbf{R}^g$. Suppose that $J = [M'M^{-1}]$, and let φ be the maximal extension of the corresponding canonical trace. Suppose that φ is defined on the two-sided ideal $\mathfrak{m} \subset \mathbf{R}^g$. An operator $T \in \mathfrak{m}^{\frac{1}{2}}$ is of the form U_a, with a left bounded and $a \in D_M$, if and only if $[TM'^{-1}]$ exists and is square integrable with respect to φ.*

Proof: (i) Suppose that $T = U_a$, where a is left bounded. Let $M = \int_0^\infty \lambda\, dE_\lambda$ and we put $\Delta_n = \left(\frac{1}{n} \leqq \lambda < +\infty\right)$ for $n = 1, 2, \ldots$. Since $M^{-1}(\Delta_n) \in \mathbf{R}^d$, and it commutes with J, $M^{-1}(\Delta_n)a$ is left bounded too, and $U_{M^{-1}(\Delta_n)a} = U_a M'^{-1}(\Delta_n)$.[12]) So we get

$$\|MM^{-1}(\Delta_n)a\|^2 = \|(I - E_{\frac{1}{n}})a\|^2 = \varphi(U_{M^{-1}(\Delta_n)a}U^*_{M^{-1}(\Delta_n)a}) =$$

$$= \varphi(TM'^{-1}(\Delta_n)[TM'^{-1}(\Delta_n)]^*) = \sum_{\alpha\in F}\|M'^{-1}(\Delta_n)T^*a_\alpha\|^2 \leqq \|a\|^2$$

for $n = 1, 2, \ldots$, provided that $\varphi(A) = \sum_{\alpha\in F}(Aa_\alpha, a_\alpha)$ for $A \in \mathfrak{m}$. This gives that $a_\alpha \in D_{M'^{-1}T^*}$, therefore $M'^{-1}T^*$ is densely defined. Since $(M'^{-1}T^*)^* \supseteq TM'^{-1}$ is densely defined too, we see that $[M'^{-1}T^*]$ exists. The series $\sum_{\alpha\in F}\|M'^{-1}T^*a_\alpha\|^2$ converges, hence $[M'^{-1}T^*]$ is square integrable with rescept to φ. By a remark made in the proof of lemma 10 the same is true for $[TM'^{-1}] = [M'^{-1}T^*]^*$.

(ii) Suppose that $[TM'^{-1}]$ exists, and is square integrable with respect to φ. Putting $\Delta_n = \left(\frac{1}{n} \leqq \lambda < +\infty\right)$, $E'(\Delta_n) = SE(\Delta_n)S$ $(n = 1, 2, \ldots)$, we have by lemma 9 $TE'(\Delta_n) = U_{a_n}$ for each n, where a_n is left bounded. We prove now that the elements a_n converge to an element $a \in \mathfrak{H}_\mathbf{R}$. For this we observe first that for $n \geqq m$ $U_{E(\Delta_m)a_n} = TE'(\Delta_n)E'(\Delta_m) = U_{a_m}$[12]), hence $E(\Delta_m)a_n = a_m$. So we need only to prove that the sequence $\|a_n\|$ is bound-

[12]) If $T \in \mathbf{P}^d$ and a is left bounded, then Ta is left bounded too, and $U_{Ta} = U_a ST^*S$, also for an a not necessarily $\in D_{J^{-1}}$. (If $a \in D_{J^{-1}}$, then the problem is settled by lemme 8a in [2]). For this we have to prove that $V_x Ta = U_a ST^*Sx$ for every $x \in \mathbf{R}$. But by lemme 8b in [2] $V_x Ta = V_{ST^*Sx}a$, and by lemme 24 $V_{ST^*Sx}a = U_a ST^*Sx$, qu. e. d.

ed. But

$$\|a_n\|^2 = \|MM^{-1}(A_n)\, a_n\|^2 =$$
$$= \varphi(U_{M^{-1}(A_n)a_n} U^*_{M^{-1}(A_n)a_n}) = \varphi(TM'^{-1}(A_n)[TM'^{-1}(A_n)]^*) \leqq \|TM'^{-1}\|_2^2$$

for every n, which proves our assertion. We have further

$$V_x a = \lim_{n\to\infty} V_x a_n = \lim_{n\to\infty} U_{a_n} x = \lim_{n\to\infty} TE'(A_n)x = Tx$$

for every $x \in \mathbf{R}$, which shows that a is left bounded and $U_a = T$. Finally we prove that $a \in D_M$. To see this, observe that

$$\|M(A_n)a\|^2 = \|Ma_n\|^2 = \varphi(U_{a_n} U^*_{a_n}) \leqq \varphi(TT^*)$$

for every n, from which our assertion follows immediately.

Remark. Lemma 11 shows immediately, that the canonical trace in a unitary algebra constructed with $M = I$ is maximal. On the other side, combined with Theorem 2 it gives possibility to construct various examples of quasi-unitary algebras with a semi-finite $\mathbf{R}^g$ for which the canonical trace is not maximal by any choice of M. Indeed, we can proceed as follows. We choose a semi-finite ring $\mathbf{N}$, a maximal trace φ with the corresponding two-sided ideal $\mathfrak{m} \subseteqq \mathbf{N}$, and a positive, self-adjoint, non-singular $H \eta \mathbf{N}$. For a positive, self-adjoint, non-singular $C \eta \mathbf{N} \cap \mathbf{N}'$ we put $H_C = [CH]$. Next we consider the two-sided ideal $\mathfrak{a}$ formed by the operators $\sum_{i=1}^{n} A_i B_i^*$, where $A_i, B_i \in \mathfrak{m}^{\frac{1}{2}}$ are such that the operators $[A_i H_C^{-1}], [B_i H_C^{-1}]$ $(i = 1, 2, \ldots, n)$ are square integrable with respect to φ. Let $\mathbf{R}$ be the quasi-unitary algebra, which corresponds by Theorem 2 to $\mathbf{N}$, H and φ. If $\mathfrak{a}$ is properly contained in $\mathfrak{m}$ for any choice of C, then the canonical trace in $\mathbf{R}$ is not maximal. Consider for example the ring $\mathbf{B}$ of all bounded operators in a Hilbert space $\mathfrak{H}$. It is known that φ is determined up to a constant factor, $\mathfrak{m}^{\frac{1}{2}}$ consists of the operators of the Hilbert—Schmidt class, and, whith the inner product $\varphi(AB^*)$, the totality of these operators forms a complete Hilbert space. Suppose further that H is bounded. Then it can be shown easily that $[AH^{-1}]$ exists and is square integrable if and only if $A = TH$, where $T \in \mathfrak{m}^{\frac{1}{2}}$. If there exists a two-sided ideal $\mathfrak{n}$ properly contained in $\mathfrak{m}$, such that $H^2 \in \mathfrak{n}$, then evidently the operators $\sum_{i=1}^{n} A_i H^2 B_i^*$, where $A_i, B_i \in \mathfrak{m}^{\frac{1}{2}}$, are in $\mathfrak{n}$, hence they do not constitute all operators of $\mathfrak{m}$.

7. It is known (cf. [10] Theorem 19 and Corollary 19.1) that if $\mathbf{R}$ is a unitary algebra and $a \in \mathfrak{H}_{\mathbf{R}}$, then the operator defined by $U'_a x = V_x a$ for $x \in \mathbf{R}$ has a minimal closed extension U_a, and the totality of these operators coincides with the set of square integrable operators with respect to the canonical trace (constructed with $M=I$). Let now $\mathbf{R}$ be a quasi-unitary algebra with a semi-finite $\mathbf{R}^g$. Suppose that $J=[M'M^{-1}]$, and denote by φ the maximal extension of the canonical trace, and by $\mathfrak{m}$ the corresponding two-sided ideal $\subset \mathbf{R}^g$. If, for an element $a \in \mathfrak{H}_{\mathbf{R}}$, the minimal closed extension of the operator defined by $U'_a x = V_x a$ for $x \in \mathbf{R}$ exists, we denote it by U_a again. We prove now the following

Theorem 4. *Define a unitary mapping* ψ *of* $\mathfrak{H}_{\mathbf{R}}$ *on the Hilbert space* L^2_φ *(of the square integrable operators with respect to* φ*) by the condition* $\psi(Mb)=U_b$, *where* b *is left bounded and* $\in D_M$. *Denote by* $T_a \in L^2_\varphi$ *the operator corresponding to* $a \in \mathfrak{H}_{\mathbf{R}}$. *Then* U_a *exists if and only if* $[T_a M']$ *exists, and then* $U_a = [T_a M']$.

Proof: We put $M=\int_0^\infty \lambda\, dE_\lambda$, $M_n=\int_0^n \lambda\, dE_\lambda$ $(n=1,2,\ldots)$. Since $E_\lambda \in \mathbf{P}^d$, i. e. it commutes with J, $E_\lambda x$ is left bounded and $\in D_M$ for any $x \in \mathbf{R}$. So the elements Mb are dense in $\mathfrak{H}_{\mathbf{R}}$. To show that the same is true for the operators U_b in the space L^2_φ, we observe that if $T \in \mathfrak{m}^{\frac{1}{2}}$ and if U_{x_α} $(x_\alpha \in \mathbf{R})$ converges strongly and boundedly in norm to I (cf. [7] Theorem 1), then $TU_{x_\alpha}=U_{Tx_\alpha}$ converges to T in the metric of L^2_φ[7]), and the elements Tx_α are left bounded. Therefore it suffices to remark that if $U_c \in \mathfrak{m}^{\frac{1}{2}}$ and c is left bounded, then putting $E'_n = SE_nS$ the operators $U_cE'_n = U_{E_nc}$[12]) converge to U_c in the metric of L^2_φ. So there exists a unitary mapping ψ of $\mathfrak{H}_{\mathbf{R}}$ on L^2_φ, which satisfies $\psi(Mb)=U_b$ for every left bounded $b \in D_M$. We divide the following in two steps.

(i) We prove that if U_a exists, then $[T_aM']$ exists too, and $U_a \supseteq [T_aM']$. Let $U_a = V|U_a|$ be the polar decomposition of U_a and $|U_a|=\int_0^\infty \lambda\, dF_\lambda$, $|U_a|_n=\int_0^n \lambda\, dF_\lambda$, $U_a^{(n)}=V|U_a|_n$ and $M'_n=SM_nS$ $(n=1,2,\ldots)$. We put $\bar{F}_n = VF_nV^*$ for $n=1,2,\ldots$, then for $x \in \mathbf{R}$

$$U_a^{(n)}x=\bar{F}_nU_ax=\bar{F}_nV_xa=V_x\bar{F}_na,$$

hence $\bar{F}_na$ is left bounded, and $U_{\bar{F}_na}=U_a^{(n)}$. We show next that

$$U_{\bar{F}_nE_ma}=[\bar{F}_nT_aM'_m]$$

for $n,m=1,2,\ldots$. By the reasoning of lemma 10, $[AT]$ and $[TA]$ exist for $A \in \mathbf{R}^g$ and $T \in L^2_\varphi$, and they are in L^2_φ. We see easily that $\|[AT]\|_2 \leqq \|A\|\,\|T\|_2$ and $\|TA\|_2=\|[A^*T^*]\|_2 \leqq \|A\|\,\|T^*\|_2=\|A\|\,\|T\|_2$. From this it follows at once, that for every $A \in \mathbf{R}^g$ there exist two bounded operators

L_A and R_A in L_φ^2, such that $L_A T = [AT]$, $R_A T = TA$ for $T \in L_\varphi^2$. We have

$$\psi(M_m M b) = \psi(M M_m b) = U_{M_m b} = U_b M'_m = R_{M'_m} U_b,$$

and

$$\psi(\bar{F}_n M b) = \psi(M \bar{F}_n b) = U_{\bar{F}_n b} = \bar{F}_n U_b = L_{\bar{F}_n} U_b.$$

Since the elements Mb (resp. U_b) are dense in $\mathfrak{H}_{\mathbf{R}}$ (resp. L_φ^2), we have, for every $c \in \mathfrak{H}_{\mathbf{R}}$, $\psi(M_m c) = R_{M'_m} T_c$ and $\psi(\bar{F}_n c) = L_{\bar{F}_n} T_c$, hence

$$\psi(M_m \bar{F}_n a) = \psi(M \bar{F}_n E_m a) = U_{E_m \bar{F}_n a} = R_{M'_m} L_{\bar{F}_n} T_a = [\bar{F}_n T_a M'_m]$$
$$(m, n = 1, 2, \ldots).$$

Supposing that $c \in D_{T_a M'}$, we choose a sequence $c_n \in \mathfrak{H}_{\mathbf{R}}$ such that $c_n \in D_{T_a M'_n}$, $E'_n c_n = c_n$ $(n = 1, 2, \ldots)$, and $\lim_{n\to\infty} c_n = c$, $\lim_{n\to\infty} M' c_n = M'c$. Since $U_{E_n \bar{F}_m a} = U_a^{(m)} E'_n$, therefore

$$U_a^{(m)} c = \lim_{n\to\infty} U_a^{(m)} c_n = \lim_{n\to\infty} U_{\bar{F}_n E_m a} c_n = \lim_{n\to\infty} \bar{F}_m T_a M'_n c_n = \lim_{n\to\infty} \bar{F}_m T_a M' c_n =$$
$$= F_m T_a M'c.$$

Hence $\lim_{m\to\infty} U_a^{(m)} c = \lim_{m\to\infty} F_m T_a M'c$, which gives at once $c \in D_{U_a}$ and $U_a c = T_a M'c$, consequently $U_a \supseteq [T_a M']$ [13]).

(ii) We prove now conversely, that if $[T_a M']$ exists, then U_a exists too, and $[T_a M'] \supseteq U_a$. We put $T_a = V|T_a|$, $|T_a| = \int_0^\infty \lambda \, dG_\lambda$, $T_a^{(n)} = V|T_a|_n$, $\bar{G}_n = V G_n V^*$ $(n = 1, 2, \ldots)$. Similarly, as in (i), we get $\psi(G_n M_m a) = T_a^{(n)} M'_m$ $(m, n = 1, 2, \ldots)$. By lemma 11 $T_a^{(n)} M'_m = U_{a'}$, where $E_m a' = a'$ and a' is left bounded. Since $\psi(Ma') = T_a^{(n)} M'_m$, we have necessarily $Ma' = M_m \bar{G}_n a = M \bar{G}_n E_m a$, which shows that $\bar{G}_n E_m a$ is left bounded, and $U_{\bar{G}_n E_m a} = T_a^{(n)} M'_m$ $(n, m = 1, 2, \ldots)$. If $x \in \mathbf{R}$ we have

$$\lim_{n\to\infty} V_x \bar{F}_n E_m a = \lim_{n\to\infty} T_a^{(n)} M'_m x,$$

therefore $M'_m x \in D_{T_a}$ and $V_x E_m a = T_a M'_m x$. Since $T_a M'_m \supseteq [T_a M'] E'_m$, we have further

$$V_x a = \lim_{m\to\infty} V_x E_m a = \lim_{m\to\infty} [T_a M'] E'_m x,$$

from which it follows that $x \in D_{[T_a M']}$, and that $V_x a = [T_a M'] x$. But this gives clearly our statement.

Putting together the two parts, if U_a exists then by (i) $[T_a M']$ exists too, and $U_a \supseteq [T_a M']$, but by (ii) $[T_a M'] \subseteq U_a$, hence $[T_a M'] = U_a$, and conversely.

So the proof of the Theorem 4 is completed.

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13) Note that $T_a M'$ is densely defined, since $T_a M'_n$ is densely defined for $n = 1, 2, \ldots$.

8. In this section we give a new proof for a theorem of DIXMIER concerning quasi-central elements, which leads to somewhat more general result (cf. [2] Theorème 4).[14]) We recall that if $\mathbf{R}$ is a quasi-unitary algebra, an element $a \in \mathfrak{H}_{\mathbf{R}}$ is quasi-central, if for every $x \in \mathbf{R}$ we have $U_{xj} a = V_x a$. One verifies easily, that the set of these elements is a closed subspace of $\mathfrak{H}_{\mathbf{R}}$, and denoting by $\mathfrak{M}$ the minimal subspace with a projection $P \in \mathbf{R}^g$ containing it, we have $P \in \mathbf{R}^g \cap \mathbf{R}^d$ too (for these cf. [2] VII).

Theorem 5. *$\mathbf{R}^g$ is of finite class if and only if $\mathfrak{M} = \mathfrak{H}_{\mathbf{R}}$.*

Proof. The proof for the sufficiency of this condition is as in DIXMIER's paper. If G is quasi-central, then we have for $x, y \in \mathbf{R}$:

$$(U_x U_y a, a) = (U_y a, U_{xjs} a) = (U_y a, V_{x^s} a) = (U_y V_{x^{sjs}} a, a) = (U_y U_x a, a).$$

From this it follows by continuity $(STa, a) = (TSa, a)$ for $S, T \in \mathbf{R}^g$. Since (Ta, a) $(T \in \mathbf{R}^g)$ is a positive linear form, it determines a trace defined on every element of $\mathbf{R}^g$. If, for $T \in \mathbf{R}^g$, $(T^* Ta, a) = 0$ for every quasi-central a, then by $\mathfrak{M} = \mathfrak{H}_{\mathbf{R}}$ we have necessarily $T = 0$. Therefore $\mathbf{R}^g$ has a complete system of traces, so that ([5] lemme 12) $\mathbf{R}^g$ is of finite class.

Conversely, suppose that $\mathbf{R}^g$ is of finite class, and $\mathfrak{M} \neq \mathfrak{H}_{\mathbf{R}}$. Suppose that $J = [M' M^{-1}], M \eta \mathbf{R}^d, M' = SMS$ (cf. Theorem 1). By central decomposition ([2] III) we can reduce the problem to the case, when $\mathfrak{M} = 0$ and the canonical trace φ is everywhere defined on $\mathbf{R}^g$. Since now $I \in L^2_\varphi$, there exists an element $a \in \mathfrak{H}_{\mathbf{R}}$ such that $T_a = I$, hence by Theorem 4 U_a exists, and $U_a = M'$. We put

$$M' = \int_0^\infty \lambda \, dE'_\lambda \quad \text{and} \quad M'_n = \int_0^n \lambda \, dE'_n \qquad (n = 1, 2, \ldots).$$

Then $U_{E'_n a} = M'_n$, hence $SJ^{-1} E'_n a = E'_n a$ for $n = 1, 2, \ldots$, and so $SJ^{-1} a = a$.

Observe now, that if $c \in D_J$ then for $x \in \mathbf{R}$ we have $J^{-1} V_{xj} Jc = V_x c$. To prove this, we choose a sequence $y_n \in \mathbf{R}$ $(n = 1, 2, \ldots)$ such that $\lim_{n \to \infty} y_n = c$ and $\lim_{n \to \infty} J y_n = Jc$. In this case

$$V_x c = \lim_{n \to \infty} V_x y_n = \lim_{n \to \infty} J^{-1} V_{xj} J y_n,$$

which shows that $V_{xj} Jc \in D_{J^{-1}}$ and $J^{-1} V_{xj} Jc = V_x c$.

We have now for $x \in \mathbf{R}$:

$$V_x a = M' x = M' J^{-1} J x = J^{-1} M J x = J^{-1} V_{xj} a = V_x J^{-1} a,$$

[14]) $\mathbf{Q}^d \subseteq \mathbf{P}^d$ is proved in our Theorem 1, and the axiom A'·5, loc. cit., will not be used in the sequel.

hence $J^{-1}a = a$, and so $Sa = a$. We have also

$$V_{xj^{-1}}a = M'J^{-1}x = Mx = SM'x^s = SV_{x^s}SSa = U_x a.$$

This shows that a is quasi-central, and since obviously $a \neq 0$, we have a contradiction, and therefore our theorem is proved.

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