On weakly complemented lattices.

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1. Introduction. It is known that for any congruence relation K of a lattice L with least element O (similarly as for any one of a ring) the set S of elements L congruent to O is an ideal of L [**2**, p. 21] which is called the *kernel* of the congruence relation K. But whereas in a ring every congruence relation is determined by its kernel, this is not in general true in the case of lattices with least element [**2**, p. 21].

G. BIRKHOFF has proposed the very interesting problem to find necessary and sufficient conditions, securing that the correspondence between the congruence relations and ideals (as kernels of the congruence relations) of a lattice with least element be one-to-one [**2**, p. 161][1]). This problem is still unsolved in general, but for distributive lattices it has been solved two years ago by AREŠKIN [**1**]. In order to characterize the distributive lattices with least element for which the correspondence between congruence relations and their kernels is one-to-one, he has introduced the concept of „weakly complemented" lattices in the following sense:

Definition 1. *A lattice L with least element O is called weakly complemented if for each pair of distinct elements u, v of L there exists at least one element x in L such that*

$$x \cap (u \cap v) = O, \quad x \cap (u \cup v) > O.$$

In accordance with the result of AREŠKIN [**1**, p. 486], weakly complemented lattices form a very important class of lattices; nevertheless, the properties of such lattices have not yet been studied. In this paper we shall be concerned just with such investigations: in section 2, we give a characterization of weakly complemented lattices, more simple than definition 1; in section 3, we discuss connections between the class of weakly complemented lattices and some other important classes of lattices.

[1]) At that time it was only known, on basis of STONE's theorem on the one-to-one correspondence between Boolean algebras and idempotent rings with unit [**4**], that for distributive lattices with least element complementedness is a sufficient condition.

2. New characterization of weakly complemented lattices. In this section we give, using the term „semi-complement", a very simple characterization for weakly complemented lattices. The term „semi-complement" was introduced in a previous paper [**5**, p. 42] by the author, but it seems useful somewhat to modify the original definition in the following manner:

Definition 2. *By the semi-complement of an element x of a lattice L with least element O is meant an element y of L such that $x \cap y = O$.*

One sees immediately that O is a trivial semi-complement for all $x (\in L)$. A semi-complement y of the element x distinct from O will be called *proper semi-complement* of x [2]).

We recall also the following

Definition 3. *A lattice L with least element is called semi-complemented if all its elements distinct from the (eventually existing) greatest element of L have proper semi-complements in L.*

Now we prove

Theorem 1. *A lattice L with least element O is weakly complemented if and only if for each pair u, v of L such that $u < v$, there exists at least one semi-complement x of u which is not a semi-complement of v* (i. e., that $u \cap x = O$, but $v \cap x > O$).

Proof. First, let L be a weakly complemented lattice with the least element O, and let u, v be any pair of L such that $u < v$. Then, by definition 1, there exists at least one element x in L such that

$$x \cap (u \cap v) = O, \quad x \cap (u \cup v) > O.$$

Hence, by $u < v$, it follows

$$x \cap u = O, \quad x \cap v > O,$$

which proves the necessity of the condition of theorem 1.

Conversely, let L be a lattice with the least element O in which the condition of theorem 1 holds, and let u, v be any pair of distinct elements of L. Since, by $u \neq v$,

$$u \cap v < u \cup v,$$

it follows by our assumptions that there exists an element x in L for which

$$x \cap (u \cap v) = O, \quad x \cap (u \cup v) > O.$$

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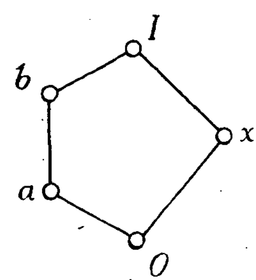
[2]) That is, the difference between the original definition and the new one is only that from now on we consider the element O as a (trivial) semi-complement of all elements of any lattice with least element O. For the advantage of this modification it suffices to remark that, by the new definition, if the element $x (\in L)$ is a semi-complement of $a (\in L)$ and y is an arbitrary element of L such that $y \leqq x$, then y is also a semi-complement of a; moreover, for any distributive lattice L with least element, the set of all semi-complements of an arbitrary element is an ideal of L.

This means just that the lattice L is weakly complemented, so that also the sufficiency of the condition of theorem 1 is proved.

Consequently, weakly complemented lattices may be defined also as follows:

Definition 4. *A lattice L with least element O is called weakly complemented if for each pair u, v of its elements such that $u < v$, there exists at least one semi-complement of u which is not a semi-complement of v.*

3. Further remarks on weakly complemented lattices. It is easily seen that *neither complementedness implies weakly complementedness*[3]) *nor conversely*. On the one hand, the lattice given by the diagramm



is obviously complemented, but, since the elements a and b have only common semi-complements (namely, the elements O and x), by definition 4 it is not weakly complemented. On the other hand, one sees easily that the family of all finite subsets of a countable set, partially ordered by set-inclusion, forms a weakly complemented lattice which is not complemented.

However, we prove

Theorem 2. *Any weakly complemented lattice is semi-complemented.*

Proof. Let L be any weakly complemented lattice with least element O and let a be any element of L not equal to the (eventually existing) greatest element of L. Then there exists an element b in L such that $a < b$. Hence, by definition 4, there exists a semi-complement x of a which satisfies the inequality

$$b \cap x > 0.$$

Consequently, $x \neq O$; that is, x is a proper semi-complement of a. Hence, by definition 3, L is semi-complemented.

On the other hand we prove

Theorem 3. *Any relatively complemented lattice with least element is weakly complemented.*

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[3]) Hence, the term "weakly complemented" is not perfectly suggestive. Really, the class of weakly complemented lattices is a generalization (not of the complemented, but) of the relatively complemented lattices. (See theorem 3 below.)

Corollary. *Any complemented modular lattice is weakly complemented.*

Proof. Let L be any relatively complemented lattice with least element O and let a, b be any pair of its elements such that $a < b$. Further, let x denote any relative complement of a in the closed interval $[O, b]$. Then, by the definition of the relative complement,

(1) $$O \leqq x \leqq b,$$

(2) $$x \cap a = O,$$

(3) $$x \cup a = b.$$

Equation (2) shows that x is a semi-complement of a. Therefore, by definition 4, it suffices to show that x is not a semi-complement of b. But, (1) implies

(4) $$x \cap b = x,$$

and (3) implies, by $a \neq b$,

(5) $$x > O.$$

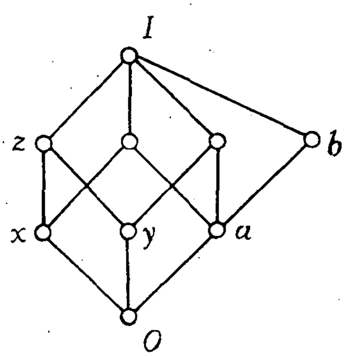
Thus, by (4) and (5), we have

$$x \cap b > O,$$

completing the proof of the theorem.

For the corollary, it suffices to recall the well-known result that any complemented modular lattice is relatively complemented [3, p. 7].

We remark that, in the corollary, the condition of modularity cannot be replaced by the weaker condition of semi-modularity. For, the lattice given by the diagramm



is complemented and semi-modular, but it is not weakly complemented, because every semi-complement of a (namely, each of the elements O, x, y, z) is also a semi-complement of b.

References.

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