Generalization of a theorem of Alexandroff and Urysohn.

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ALEXANDROFF and URYSOHN [1] proved the following theorem. If to every ordinal number α of the second class there corresponds an ordinal number $\mu(\alpha)$ such that $\mu(\alpha) < \alpha$, then there exists a non-denumerable set of ordinal numbers α of the second class such that the corresponding $\mu(\alpha)$ are all equal. BEN DUSHNIK [2] proved the following more general result. If to every ordinal number $\alpha < \omega_{\nu+1}$ there corresponds an ordinal number $\mu(\alpha)$ such that $\mu(\alpha) < \alpha$, then there exists an ordinal number β such that the equation $\mu(\alpha) = \beta$ has $\aleph_{\nu+1}$ solutions. ERDŐS [3] proved that the following generalisation holds. If ω_ν is not confinal to ω and to every ordinal number $\alpha < \omega_\nu$ there corresponds an ordinal number $\mu(\alpha)$ such that $\mu(\alpha) < \alpha$, then there exists an ordinal number $\beta < \omega_\nu$ and a subset N of $W(\omega_\nu) = \{\alpha : \alpha < \omega_\nu\}$ such that N is confinal to $W(\omega_\nu)$ and $\mu(\alpha) \leqq \beta$ for every $\alpha \in N$. NOVÁK [4] proved the following theorem. If M is a closed subset of the type ω_1 of $W(\omega_1)$ and to every element $\alpha \in M$ there corresponds an ordinal number $\mu(\alpha)$ such that $\mu(\alpha) < \alpha$, then there exists a non-denumerable set of ordinal numbers $\alpha \in M$ such that the corresponding $\mu(\alpha)$ are all equal. NEUMER [5] proved the following more general result. Let $\Lambda > \omega$ be a regular ordinal number of the second kind and M a subset of $W(\Lambda)$. If $W(\Lambda) - M$ does not contain a closed subset similar to $W(\Lambda)$ and to every element $\alpha \in M$ there corresponds an ordinal number $\mu(\alpha)$ such that $\mu(\alpha) < \alpha$ then there exists a set of ordinal numbers $\alpha \in M$ similar to $W(\Lambda)$ such that the corresponding $\mu(\alpha)$ are all equal.

In the present paper we shall prove by a modification of the method of DUSHNIK and NOVÁK a theorem which contains all the preceding theorems.

T h e o r e m. *Let Λ be an ordinal number of the second kind which is not confinal to ω and M a subset of $W(\Lambda) = \{\alpha : \alpha < \Lambda\}$. Suppose that to every element $\alpha \in M$ there corresponds an ordinal number $f(\alpha)$ such that $f(\alpha) < \alpha$. If $W(\Lambda) - M$ does not contain a closed subset confinal to $W(\Lambda)$*[1]*, then there exists an ordinal number $\pi < \Lambda$ and a subset N of M such that N is confinal to $W(\Lambda)$ and $f(\alpha) \leqq \pi$ for every $\alpha \in N$.*

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[1]) A subset R of $W(\Lambda)$ is called closed if the limit of any fundamental sequence of elements of R belongs to R whenewer this limit is smaller than Λ. We call a subset S of $W(\Lambda)$ confinal to $W(\Lambda)$ if for every $\nu \in W(\Lambda)$ there is a $\mu \in S$ such that $\mu > \nu$.

Proof. Put $D=\{f(\alpha)\}_{\alpha\in M}\cap(W(\Lambda)-M)$. We shall consider two cases:

1) the set D is not confinal to $W(\Lambda)$,

2) the set D is confinal to $W(\Lambda)$.

Ad 1): Since $f(\alpha)<\alpha$ for $\alpha\in M$, to every element $\alpha\in M$ there corresponds a natural number $n(\alpha)$ such that

$$f_{(\alpha)}^{(n(\alpha))}\in D.^{2)}$$

Let Φ be the smallest μ for which Λ is confinal to μ, and M' a subset of the type Φ of M confinal to $W(\Lambda)$. As Λ is not confinal to ω, therefore there exists a natural number m and a subset $\{\gamma_\eta\}$ of M' such that

$$\overline{\overline{\{\gamma_\eta\}}}=\overline{\Phi} \text{ and } f_{(\gamma_\eta)}^{(m)}\in D.$$

It is obvious that $\{\gamma_\eta\}$ is confinal to $W(\Lambda)$. Consider now the sequence

$$f_{(\gamma_0)}^{(m)},\ f_{(\gamma_1)}^{(m)},\ f_{(\gamma_2)}^{(m)},\ \ldots.$$

By the condition D is not confinal to $W(\Lambda)$. The set $\{f_{(\gamma_\eta)}^{(m)}\}$ is not confinal to $W(\Lambda)$, because $\{f_{(\gamma_\eta)}^{(m)}\}\subseteqq D$. It follows that there exists a smallest natural number $m_0\leqq m$ such that the set $\{f_{(\gamma_\eta)}^{(m_0)}\}$ is not confinal to $W(\Lambda)$. Let $N=\{f_{(\gamma_\eta)}^{(m_0-1)}\}$ and π the smallest ordinal numbers β for which $\beta>f_{(\gamma_\eta)}^{(m_0)}$, for every $\eta<\Phi$.

Ad 2). Suppose that the theorem is false. Then to every $\alpha\in M$ there exists an ordinal number $\varphi(\alpha)<\Lambda$ such that for $\alpha'\geqq\varphi(\alpha)$, $f(\alpha')>\alpha$. Let now M' be the set of those $\alpha\in M$, for which $f(\alpha)\notin M$. Since D is confinal to $W(\Lambda)$ and $f(\alpha)<\alpha$ for every $\alpha\in M$, therefore the set M' is confinal to $W(\Lambda)$. We define by transfinite induction a set $\{\alpha_\eta\}$ of elements $\alpha\in M'$, arranged in their natural order, in the following manner. Let $\alpha_0(\geqq 1)$ be an arbitrary element of M' and suppose that the elements $\alpha_\eta\in M'$ are defined for every $\eta<\beta(<\Lambda)$ such that

$$f(\alpha_\eta)>\alpha_\varkappa$$

for every $\varkappa<\eta$. If there is an ordinal number $\alpha\in M$ such that $\alpha_\eta<\alpha$ for every $\eta<\beta$, then let α' be the smallest such ordinal number, if β is an ordinal number of second kind and let $\alpha'=\alpha_\nu$ if $\beta=\nu+1$. We define α_β as the smallest $\alpha\in M'$ for which $\alpha\geqq\varphi(\alpha')$. In the opposite case we do not define α_β. By the definition the set $\{\alpha_\eta\}$ is confinal to $W(\Lambda)$.

Since $W(\Lambda)-M$ does not contain any closed subset confinal to $W(\Lambda)$, there exists an ordinal number of the second kind $\varkappa$ and a fundamental subsequence $\{\alpha_{\eta_\xi}\}_{\xi<\varkappa}$ of type $\varkappa$ of $\{\alpha_\eta\}$ such that

$$\lim_{\xi<\varkappa} f(\alpha_{\eta_\xi})\in M.$$

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2) If $f_{(\alpha)}^{(n-1)}\in M$, then $f_{(\alpha)}^{(n)}$ is the ordinal number corresponding to $f_{(\alpha)}^{(n-1)}$.

Put $\alpha^* = \lim_{\xi<\varkappa} f(\alpha_{\eta_\xi})$. It is obvious that

$$\alpha^* = \lim_{\xi<\varkappa} \alpha_{\eta_\xi}$$

since by the definition of $\{\alpha_\eta\}$

$$\alpha_0 < f(\alpha_1) < \alpha_1 < f(\alpha_2) < \alpha_2, \dots .$$

Since $f(\alpha^*) < \alpha^*$, there exists an ordinal number ξ_0 such that

$$f(\alpha^*) < \alpha_{\eta_{\xi_0}}.$$

This is impossible, since $\alpha^* > \alpha_{\eta_{\xi_0}+1} \geqq \varphi(\alpha_{\eta_{\xi_0}})$ and by the definition $f(\alpha^*) > \alpha_{\eta_{\xi_0}}$. The theorem is proved.

Remark. The theorem can not be improved by weakening the assumption that $W(\Lambda)-M$ does not contain a closed subset confinal to $W(\Lambda)$. (See the proof of theorem 4 of [5].)

Added in proof. BACHMANN [6] proved the following theorem. Let Λ be an ordinal number of the second kind which is not confinal to ω and M a closed subset of $W(\Lambda)$ similar to $W(\Lambda)$. If to every $\alpha \in M$ there corresponds an ordinal number $\mu(\alpha)$ such that $\mu(\alpha) < \alpha$, then there exists an ordinal number $\beta < \Lambda$ and a set N of ordinal number $\alpha \in M$ such that N is confinal to $W(\Lambda)$ and $\mu(\alpha) \leqq \beta$, for every $\alpha \in N$. (This is contained in our theorem. See the theorem 2 of § 7 and theorem 3 of § 9 in [6].)

References.

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[6] H. BACHMANN, *Transfinite Zahlen* (Ergebnisse der Math. und ihrer Grenzgebiete, Neue Folge, Heft **1**, Berlin—Heidelberg—Göttingen, 1955), S. 43.

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