Some results concerning a problem in set theory.

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Let S be a given set of power $\mathfrak{m} \geqq \aleph_0$, Q a subset of power $\mathfrak{q} \geqq \aleph_0$ of S and suppose that to every element x of Q there corresponds a subset $H(x)$ of S such that for any $x \in Q$ the power of the set $H(x)$ is smaller than a given cardinal number $\mathfrak{n}$ which is smaller than $\mathfrak{m}$ and $\overline{\overline{\bigcup_{x\in Q} H(x)}} = \mathfrak{q}$. Let $\mathfrak{p}$ be a cardinal number for which $\mathfrak{p} \leqq \mathfrak{m}$.

Definition. *A subset Γ of Q is said to have the property $T(\mathfrak{q}, \mathfrak{p})$ whenever*

$$1)\ \overline{\overline{\bigcup_{x\in\Gamma} H(x)}} = \mathfrak{q} \quad \textit{and} \quad 2)\ \overline{\overline{\bigcup_{\substack{x,y\in\Gamma\\ x\neq y}} (H(x)\cap H(y))}} < \mathfrak{p}.$$

Problem. *Does there always exist a subset Γ of Q with the property $T(\mathfrak{q}, \mathfrak{p})$, if $\mathfrak{q} > \mathfrak{n}$, $\mathfrak{p} \geqq \mathfrak{n}$ and $\mathfrak{q} \geqq \mathfrak{p}$?*

We shall prove in this paper that the answer to this problem is affirmative in the following cases:

a) if $\mathfrak{p} = \mathfrak{q}$ (in the case, when $\mathfrak{q}\ (\neq \aleph_{\alpha+\omega})$ is the sum of $\mathfrak{n}$ cardinal numbers, each of which is smaller than $\mathfrak{q}$, we assume the generalized continuum hypothesis) (Theorem 1, Theorem 6 and Theorem 8),

b) if $\mathfrak{q} = \aleph_{\alpha+\omega}$ (where α is an arbitrary and ω the smallest infinite ordinal number) and $\mathfrak{p} = \aleph_{\alpha+1}$ (Theorem 6),

c) if 1) $\mathfrak{n} = \mathfrak{p} = \aleph_0$ and $\mathfrak{q}$ is a regular cardinal number or if 2) $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, $\mathfrak{q} = \aleph_{\alpha+2}$ (where α is an arbitrary ordinal number) and $\mathfrak{p} = \mathfrak{n} \leqq \aleph_\alpha$ (Theorem 7),

d) if $\mathfrak{q}$ is a singular cardinal number and if 1) $\mathfrak{n} = \aleph_0$, $\mathfrak{p} = [\mathfrak{q}^* \cdot \mathfrak{n}]^+$, or if 2) $\aleph_0 \leqq \mathfrak{n} < \mathfrak{q}$, $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for every $\aleph_\alpha$ for which $\aleph_\alpha < \mathfrak{q}$ and $\mathfrak{p} = [\mathfrak{q}^* \cdot \mathfrak{n}]^+$ (where $\mathfrak{q}^*$ denotes the smallest cardinal number such that $\mathfrak{q}$ is the sum of $\mathfrak{q}^*$ cardinal numbers each of which is less than $\mathfrak{q}$, and $\mathfrak{r}^+$ the cardinal number immediately following $\mathfrak{r} = \mathfrak{q}^* . \mathfrak{n}$) (Theorem 8),

e) in every case whenever $\mathfrak{q} = \aleph_\alpha$ is a singular cardinal number such that α is not confinal to ω and $\mathfrak{q} = \mathfrak{p}$, supposing that the answer to the problem is affirmative in the special case whenever $\mathfrak{q} = \mathfrak{p} = \aleph_\beta$ and β is confinal to ω (Theorem 11).

The answer to the problem is negative, in general, in the following cases. These results are due to P. ERDŐS.

a) If $\mathfrak{q}$ is a singular cardinal number and $\mathfrak{p}=(\mathfrak{q}^*)^+$ (Theorem 9).

b) If $\mathfrak{q}=\aleph_{\alpha+1}$, $\aleph_\alpha=\mathfrak{r}$ is singular, $\mathfrak{p}\leqq\mathfrak{n}$, $\mathfrak{n}=(\mathfrak{r}^*)^+$ and $2^{\aleph_\beta}=\aleph_{\beta+1}$ for every β (Theorem 10).

Remark. If the answer to the problem is affirmative with $\mathfrak{q}=\mathfrak{p}$ then the answer to the following problem of RUZIEWICZ [1] is affirmative with $\mathfrak{c}>\aleph_1$ too:

Let E be a given non countable set of power $\mathfrak{c}$ and suppose that there exists a relation R between the elements of E such that for any $x\in E$, the power of the set of the elements $y\in E$ $(y\neq x)$ for which xRy holds, is smaller than a given cardinal number $\mathfrak{r}\geqq\aleph_0$ which is smaller than $\mathfrak{c}$. Two distinct elements x and y of E are called independent if neither xRy nor yRx. We say that a subset of E is a free set if any two points of this subset are independent.

Problem of Ruziewicz. *Does there always exist a free subset of power $\mathfrak{c}$ of E?*

The answer to this problem is affirmative first if $\mathfrak{r}=\aleph_0$ and $\mathfrak{c}$ is either of the form $2^{\mathfrak{f}}$ or of the form $\aleph_{\alpha+1}$ ([2], [3]), then if $\mathfrak{c}$ is a regular cardinal number or if $\mathfrak{c}$ is the countable sum of cardinals smaller than $\mathfrak{c}$ ([4], [5]), finally, in the general case, assuming the generalized continuum hypothesis [6].

We shall consider two cases: 1) there exists a regular cardinal number $\mathfrak{s}$ for which $\mathfrak{r}\leqq\mathfrak{s}<\mathfrak{c}$, 2) there is no such a regular cardinal number $\mathfrak{s}$. It is obvious that in the second case $\mathfrak{c}$ is regular and $\mathfrak{r}$ singular. Thus there exists in this case a regular cardinal number $\mathfrak{r}_0<\mathfrak{r}$ and a subset F of power $\mathfrak{c}$ of E such that, for every $x\in F$, the power of the set $R(x)$ of the elements $y\in E$ for which xRy holds, is smaller than $\mathfrak{r}_0$. [Indeed let $\mathfrak{r}^*$ denote the smallest cardinal number such that $\mathfrak{r}$ is the sum of $\mathfrak{r}^*$ cardinal numbers each of which is less than $\mathfrak{r}$. Since $\mathfrak{r}$ is singular, we have $\mathfrak{r}^*<\mathfrak{r}$. Let $\varphi_{\mathfrak{r}^*}$ denote the initial number of $\mathfrak{r}^*$. There exist regular cardinal numbers $\mathfrak{r}_1, \mathfrak{r}_2, \ldots, \mathfrak{r}_\xi, \ldots$ $(\xi<\varphi_{\mathfrak{r}^*})$ such that $\mathfrak{r}_\alpha>\mathfrak{r}_\beta>\mathfrak{r}^*$ for $\alpha>\beta$ and

$$\mathfrak{r}=\mathfrak{r}_1+\mathfrak{r}_2+\cdots+\mathfrak{r}_\xi+\cdots.$$

Let E_ξ be the set of elements $x\in E$, for which $\overline{\overline{R(x)}}<\mathfrak{r}_\xi$. Obviously $\bigcup\limits_{\xi<\varphi_{\mathfrak{r}^*}} E_\xi=E$. As $\mathfrak{c}$ is regular and $\mathfrak{r}<\mathfrak{c}$, therefore there exists an ordinal number $\xi_0(<\varphi_{\mathfrak{r}^*})$ such that $\overline{\overline{E_{\xi_0}}}=\mathfrak{c}$. Let $F=E_{\xi_0}$ and $\mathfrak{r}_0=\mathfrak{r}_{\xi_0}$.]

Define now the relation R' as follows: Let $xR'y$ if there exists a sequence $x_1, x_2, \ldots, x_k$ of elements of E such that xRx_1, $x_1Rx_2, \ldots, x_kRy$ hold. It is obvious that R' is transitive. In the first case, for any $x\in E$, the power of the set $R'(x)$ of elements $y\in E$, for which $xR'y$ holds, is smaller than $\mathfrak{s}_0<\mathfrak{c}$, where $\mathfrak{s}_0$ is a given regular cardinal number such that $\mathfrak{r} \leqq \mathfrak{s}_0 < \mathfrak{e}$. If $\mathfrak{r} = \aleph_0$, then we suppose even that $\mathfrak{s}_0 > \aleph_0$; this can be done, since $\mathfrak{e} > \aleph_1$. In the second case, for any $x \in F$, the power of the set $R'(x)$ of elements $y \in E$, for which $xR'y$ holds, is smaller than $\mathfrak{r}_0 < \mathfrak{e}$. [Indeed let x be a given element of E and $R(x) = E_1$, $R(E_1) = \bigcup_{x \in E_1} R(x) = E_2, \ldots, R(E_{k-1}) = E_k, \ldots$. It can be easily seen by induction that $\overline{\overline{E}}_k < \mathfrak{b}$ $(k = 1, 2, \ldots)$, since $\mathfrak{b}$ is regular, where $\mathfrak{b} = \mathfrak{s}_0$ in the first case and $\mathfrak{b} = \mathfrak{r}_0$ in the second case with $x \in F$. Obviously $R'(x) = \bigcup_{k < \infty} E_k$ and $\overline{\overline{\bigcup_{k<\infty} E_k}} < \mathfrak{b}$.] Let $P(x) = \{x\} \cup \{y \in E : xR'y\}$. The conditions of the problem are satisfied in the first case with $S = Q = E$, $\mathfrak{m} = \mathfrak{q} = \mathfrak{e}$, $\mathfrak{n} = \mathfrak{s}_0$, $H(x) = P(x)$, and in the second case with $S = E$, $Q = F$, $\mathfrak{m} = \mathfrak{q} = \mathfrak{e}$, $\mathfrak{n} = \mathfrak{r}_0$, $H(x) = P(x)$. If the answer to the problem is affirmative with $\mathfrak{m} = \mathfrak{q} = \mathfrak{p}$, then there exists a subset Γ of Q with the property $T(\mathfrak{m}, \mathfrak{m})$.

Let $\Sigma_\Gamma = \bigcup_{x \in \Gamma} P(x)$ and $\Pi_\Gamma = \bigcup_{x \neq y \in \Gamma} (P(x) \cap P(y))$. As $\overline{\overline{P(x)}} < \mathfrak{n} < \mathfrak{m}$, $\overline{\overline{\Sigma}}_\Gamma = \mathfrak{m} = \mathfrak{e}$ and $\overline{\overline{\Pi}}_\Gamma < \mathfrak{m} = \mathfrak{e}$, therefore there exists a subset Γ' of power $\mathfrak{m}$ of Γ such that, for any $x \in \Gamma'$, $P'(x) = P(x) - \Pi_\Gamma \neq 0$.

Let us select from every set $P'(x)$ $(x \in \Gamma')$ an element. The set of these elements is obviously free.

N o t a t i o n s. For any subset Q' of Q let

$$\Sigma_{Q'} = \bigcup_{x \in Q'} H(x),$$
$$\Pi_{Q'} = \bigcup_{\substack{x, y \in Q' \\ x \neq y}} (H(x) \cap H(y)).$$

For any cardinal number $\mathfrak{r}$ we denote by $\varphi_\mathfrak{r}$ the initial number of $\mathfrak{r}$, by $\mathfrak{r}^*$ the smallest ordinal number for which $\mathfrak{r}$ is the sum of $\mathfrak{r}^*$ cardinal numbers each of which is smaller than $\mathfrak{r}$ and by $\mathfrak{r}^+$ the cardinal number immediately following $\mathfrak{r}$. For any limes ordinal number β we denote by β^* the smallest ordinal γ for which β is confinal to γ.

T h e o r e m 1. *If* $\mathfrak{q} = \mathfrak{p}$ *and* $\mathfrak{q}$ *is not the sum of* $\mathfrak{n}$ *cardinal numbers, each of which is smaller than* $\mathfrak{q}$, *then the answer to the problem is affirmative.*

P r o o f. Assume that the theorem is false, i. e.:

(A) if M is a subset of Σ_Q for which $\overline{\overline{M}} < \mathfrak{q}$, then for every subset Γ of Q for which

$$\Pi_\Gamma \subseteq M,$$

the power of the set Σ_Γ is smaller than $\mathfrak{q}$.

It follows from the conditions $\overline{\overline{\Sigma}}_Q = q$ and $\overline{\overline{H(x)}} < \mathfrak{n} < \mathfrak{q}$ that

(B) if M is a subset of Σ_Q such that $\overline{\overline{M}} < \mathfrak{q}$, then the power of the set of elements $x \in Q$, for which

$$H(x) \cap (\Sigma_Q - M) \neq 0,$$

is $\mathfrak{q}$.

Define the sets M_β and K_β by transfinite induction as follows. Let M_0 be a subset of power less than $\mathfrak{q}$ of Σ_Q and $K_0=0$. Let now β be an ordinal number, $1\leqq\beta<\varphi_\mathfrak{n}$, and suppose that all sets M_ξ and K_ξ, where $0\leqq\xi<\beta$, have been already defined such that $M_\xi\subset\Sigma_Q$, $\overline{\overline{M}}_\xi<q$. As $\overline{\beta}<\mathfrak{n}<\mathfrak{q}$ and $\mathfrak{q}$ is not the sum of $\mathfrak{n}$ cardinal numbers, each of which is smaller than $\mathfrak{q}$, the power of the set

$$N_\beta=\bigcup_{\xi<\beta}M_\xi$$

is less than $\mathfrak{q}$. Let K_β be a set of elements $x\in Q$ such that

1) $H(x)\cap(\Sigma_Q-N_\beta)\neq0$,

2) $\Pi_{K_\beta}\subseteqq N_\beta$,

3) for every element x of $Q-K_\beta$ for which $H(x)\neq0$ there is an element $y\in K_\beta$ such that the set $H(x)\cap H(y)$ is not a subset of N_β.

Let

$$M_\beta=\Sigma_{K_\beta}-N_\beta.$$

As $N_\beta<\mathfrak{q}$ we obtain by (B) that $K_\beta\neq0$, i. e. $M_\beta\neq0$. By (A) the power of the set Σ_{K_β} is smaller than $\mathfrak{q}$. It follows that $\overline{\overline{M}}_\beta<\mathfrak{q}$. Consider the set $M=\bigcup_{\xi<\varphi_\mathfrak{n}}M_\xi$. Obviously $\overline{\overline{M}}<\mathfrak{q}$ because $\mathfrak{q}$ is not the sum of $\mathfrak{n}$ cardinal numbers, each of which is smaller than $\mathfrak{q}$ and $\overline{\varphi}_\mathfrak{n}=\mathfrak{n}<\mathfrak{q}$. It follows from (B) that there is an element x_0 of Q for which

$$H(x_0)\cap(\Sigma_Q-M)\neq0.$$

Clearly $x_0\notin K_\xi(\xi<\varphi_\mathfrak{n})$. In the opposite case there would be an ordinal number $\xi_0<\varphi_\mathfrak{n}$ such that $x_0\in K_{\xi_0}$. By the definition

$$M_{\xi_0}=\bigcup_{x\in K_{\xi_0}}H(x)-N_{\xi_0},$$

i. e.

$$H(x)\subset N_{\xi_0+1}=\bigcup_{\xi<\xi_0+1}M_\xi\subset M=\bigcup_{\xi<\varphi_\mathfrak{n}}M_\xi,$$

which is impossible.

By the property (3) of K_ξ there exists an element $y_\xi\in K_\xi$ for which the set $H(x_0)\cap H(y_\xi)$ is not a subset of N_ξ. According to the definition of M_ξ we have

$$M_\xi\cap H(x_0)\neq0$$

for every $\xi<\varphi_\mathfrak{n}$. Let $a_\xi\in M\cap H(x_0)$. Since $M_\nu\cap M_\mu=0$ if $\nu\neq\mu$, therefore the elements a_ξ are distinct. Thus the power of the set of elements a_ξ is $\mathfrak{n}$. It follows that

$$\overline{\overline{H(x_0)}}\geqq\mathfrak{n}.$$

This is impossible, since $\overline{\overline{H(x)}}<\mathfrak{n}$, for every element $x\in Q$. The theorem is proved.

Corollary. *If $\mathfrak{n} < \aleph_0$ and $\mathfrak{q} = \mathfrak{p}$, then the answer to the problem is affirmative.*

Theorem 2. *If $\mathfrak{q}$ is a regular cardinal number and there exists a subset Γ of Q with the property $T(\mathfrak{q}, \mathfrak{p})$ such that*

$$\overline{\overline{\Pi_\Gamma}} = \mathfrak{r}$$

is a regular cardinal number and $\mathfrak{r} \geqq \mathfrak{n}$, then there is a subset of Γ with the property $T(\mathfrak{q}, \mathfrak{r})$.

First we prove the following

Lemma. *Let $\mathfrak{a}$ be a regular cardinal number, A a set of power $\mathfrak{a}$ and $\mathfrak{b}$ a cardinal number, which is smaller than $\mathfrak{a}$. If to every element x of A there corresponds an ordinal number $g(x) < \varphi_\mathfrak{b}$, then there exists an ordinal number $\pi < \varphi_\mathfrak{b}$ and a subset A' of power $\mathfrak{a}$ of A such that for every element x of A' we have $g(x) < \pi$.*

Proof. Let $K(\alpha)$ denote for every ordinal number $\alpha < \varphi_\mathfrak{b}$ the set of all $x \in A$ for which $g(x) = \alpha$. It is clear that

$$A = \bigcup_{\alpha < \varphi_\mathfrak{b}} K(\alpha).$$

As $\mathfrak{b} < \mathfrak{a}$ and $\mathfrak{a}$ is regular it follows that there exists an ordinal number $\pi' < \varphi_\mathfrak{b}$ for which $\overline{\overline{K(\pi')}} = \mathfrak{a}$. By the definition of $K(\alpha)$ the lemma holds with $A' = K(\pi')$ and $\pi = \pi' + 1$.

Proof of the theorem 2. Let $\bigcup_{\gamma < \varphi_\mathfrak{r}} K_\gamma = \Pi_\Gamma$ be a decomposition of Π_Γ into the sum of mutually disjoint non empty sets K_γ $(\gamma < \varphi_\mathfrak{r})$ such that for any $\nu < \varphi_\mathfrak{r}$ we have

$$\overline{\overline{\bigcup_{\gamma < \nu} K_\gamma}} < \mathfrak{r}.$$

Consider now the set Ψ of $x \in \Gamma$ for which $H(x) - \Pi_\Gamma \neq 0$. Clearly the power of the set Ψ is $\mathfrak{q}$, because $\overline{\overline{H(x)}} < \mathfrak{n} < \mathfrak{q}$, $\overline{\overline{\Sigma_\Gamma}} = \mathfrak{q}$ and $\overline{\overline{\Pi_\Gamma}} < \mathfrak{q}$.

As $\overline{\overline{H(x)}} < \mathfrak{n}$ and $\mathfrak{r} (\geqq \mathfrak{n})$ is regular, for every $x \in \Psi$ there exists an ordinal number $f(x) < \varphi_\mathfrak{r}$ so that for any $\gamma > f(x)$, $K_\gamma \cap H(x)$ is empty. Thus by the lemma there exists an ordinal number $\pi < \varphi_\mathfrak{r}$ and a subset Ψ' of power $\mathfrak{q}$ of Ψ such that for every element x of Ψ' we have $f(x) < \pi$. If $x \in \Gamma$, then the sets $H'(x) = H(x) - \Gamma$ are mutually disjoint. It follows by the definition of Ψ that $\Sigma_{\Psi'} = \mathfrak{q}$. As

$$\Pi_{\Psi'} \subseteqq \bigcup_{\gamma < \pi} K_\gamma$$

and $\overline{\overline{\bigcup_{\lambda < \pi} K_\gamma}} < \mathfrak{r}$, therefore Ψ' has the property $T(\mathfrak{q}, \mathfrak{r})$. The theorem is proved.

Theorem 3. *If $\mathfrak{q} = \aleph_{\alpha+k}$ (where α is an arbitrary and k (> 1) a finite ordinal number) and $\mathfrak{r} = \max\{\aleph_{\alpha+1}, \mathfrak{n}^+\}$, then there is a subset of Q with the property $T(\mathfrak{q}, \mathfrak{r})$.*

Proof. Let L denote the set of cardinal numbers $\mathfrak{f}$ for each of which there exists a subset of Q with the property $T(\mathfrak{q}, \mathfrak{f})$. L is non empty, since by the theorem 1, $\aleph_{\alpha+k} \in L$. According to the well-ordering theorem L is well-ordered. Let $\mathfrak{f}_0$ be the first element of L. By the theorem 2, $\mathfrak{f}_0 \leqq \mathfrak{r}$. The theorem is proved.

Theorem 4. *Let* $\mathfrak{q}$ *be a singular cardinal number,* $\mathfrak{r}_0$ *a cardinal number which is smaller than* q *and* $\{\mathfrak{q}_\xi\}_{\xi<\varphi_{\mathfrak{q}^*}}$ *a sequence of regular cardinal numbers such that* $\mathfrak{r}_0 < \mathfrak{q}_\xi$, $\mathfrak{q}_\beta > \mathfrak{q}_\alpha (\beta > \alpha)$, $\max\{\mathfrak{n}, \mathfrak{q}^+\} < \mathfrak{q}_\xi < \mathfrak{q}$ *and* $\mathfrak{q} = \sum_{\xi<\varphi_{\mathfrak{q}^*}} \mathfrak{q}_\xi$. *If, for every* $\xi < \varphi_{\mathfrak{q}^*}$, Q_ξ *is subset of power* $\mathfrak{q}_\xi$ *of* Q *such that* Q_ξ *has a subset* Q'_ξ *with the property* $T(q_\xi, \mathfrak{r}_0)$, *then* Q *has a subset with the property* $T(\mathfrak{q}, [\mathfrak{r}_0 \mathfrak{q}^*]^+)$.

Proof. Define the sets $Q''_\xi (\xi < \varphi_{\mathfrak{q}^*})$ by transfinite induction as follows: Let $Q''_0 = Q'_0$. Let furthermore η be an ordinal number, $0 < \eta < \varphi_{\mathfrak{q}^*}$, and suppose that all sets Q''_ξ, where $0 \leqq \xi < \eta$, have been already defined such that $\overline{\overline{Q''_\xi}} = \mathfrak{q}_\xi$ and

$$\overline{\overline{\Sigma_{L_\xi} \cap \Sigma_{Q''_\xi}}} \leqq \mathfrak{r}_0$$

where $L_\xi = \bigcup_{\zeta<\xi} Q''_\zeta$.

Now we show that Q'_η has a subset R_η for which

a) $\overline{\overline{\Sigma_{R_\eta}}} < \mathfrak{q}_\eta$,

b) $\overline{\overline{\Sigma_{L_\eta} \cap \Sigma_{Q'_\eta - R_\eta}}} \leqq \mathfrak{r}_0$ (where $L_\eta = \bigcup_{\zeta<\eta} Q''_\zeta$).

It is obvious that for fixed η the sets $H^\eta(x) = H(x) - \Pi_{Q'_\eta} (x \in Q'_\eta)$ are mutually disjoint. As $\mathfrak{q}_\eta$ is a regular cardinal number, $q_\xi < q_\eta$ $(\xi < \eta)$ and $\max\{\overline{\eta}, \mathfrak{n}\} < \mathfrak{q}_\eta$, therefore

$$\overline{\overline{\bigcup_{x\in L_\eta} H(x)}} = \overline{\overline{\bigcup_{\xi<\eta} \bigcup_{x\in Q''_\xi} H(x)}} \leqq \overline{\eta}\, \mathfrak{n} \sum_{\xi<\eta} \mathfrak{q}_\xi < \overline{\eta}\, \mathfrak{n}\, \mathfrak{q}_\eta = \mathfrak{q}_\eta.$$

It follows that for the set R_η of elements x of Q'_η for which

$$H^\eta(x) \cap \left(\bigcup_{y\in L_\eta} H(y)\right) \neq 0,$$

the relation

$$\overline{\overline{\Sigma_{R_\eta}}} < q_\eta$$

holds. Let $Q''_\eta = Q'_\eta - R_\eta$. As $\overline{\overline{\Sigma_{R_\eta}}} < \mathfrak{q}_\eta$ and Q'_η has the property $T(\mathfrak{q}_\eta, \mathfrak{r}_0)$, therefore $\overline{\overline{Q''_\eta}} = \mathfrak{q}_\eta$ and Q''_η has the property $T(\mathfrak{q}_\eta, \mathfrak{r}_0)$.

Let

$$\Gamma = \bigcup_{\xi<\varphi_{\mathfrak{q}^*}} Q''_\xi.$$

Clearly Γ has the property $T(\mathfrak{q}, [\mathfrak{r}_0, \mathfrak{q}^*]^+)$.

Theorem 5. *If* $\mathfrak{q} > \aleph_0$ *and* $\mathfrak{q}_0$ *is a cardinal number such that* $\mathfrak{n} < \mathfrak{q}_0 < \mathfrak{q}$ *and* $\mathfrak{q}_0 \geqq \aleph_0$, *then there exists a subset* Q_0 *of* Q *such that*

$$\overline{\overline{Q}}_0 = \overline{\overline{\Sigma}}_{Q_0} = \mathfrak{q}_0.$$

Proof. Let $\{x_\xi\}_{\xi<\varphi_\mathfrak{q}}$ be any well-ordering of Q of type $\varphi_\mathfrak{q}$. We define a sequence of the type $\varphi_\mathfrak{q}$ of elements of Q by transfinite induction in the following way: Put $y_0 = x_0$. Let now η be an ordinal number, $0 < \eta < \varphi_\mathfrak{q}$, and suppose that all elements y_ξ, where $0 \leqq \xi < \eta$, have been already defined. Let ζ_0 be the smallest ordinal number ζ for which

$$H(x_\zeta) \not\subseteq \bigcup_{\xi<\eta} H(y_\xi). \tag{1}$$

There exists such an ordinal number ζ, for $\overline{\overline{\eta}} < \mathfrak{q}$, $\overline{\overline{H(x)}} < \mathfrak{n} < \mathfrak{q}$ and $\overline{\overline{\Sigma}}_Q = \mathfrak{q}$. Let $y_\eta = x_{\zeta_0}$. It follows from (1) that $\overline{\overline{\bigcup_{\zeta<\varphi_{\mathfrak{q}_0}} H(y_\zeta)}} = \mathfrak{q}_0$, since $\overline{\overline{H(y_\zeta)}} < \mathfrak{n} < \mathfrak{q}_0$. Let $Q_0 = \{y_\zeta\}_{\zeta<\varphi_{\mathfrak{q}_0}}$. Obviously $\overline{\overline{Q_0}} = \mathfrak{q}_0$. The theorem is proved.

Theorem 6. *If* $\mathfrak{q} = \aleph_{\alpha+\omega}$ *(where* α *is an arbitrary and* ω *the smallest infinite ordinal number) and* $\mathfrak{p} = \max\{\aleph_{\alpha+1}, \mathfrak{n}^+\}$, *then* Q *has a subset* Γ *with the property* $T(\mathfrak{q}, \mathfrak{p})$.

Proof. Let k_0 be the smallest natural number k for which $\mathfrak{n} < \aleph_{\alpha+k}$. By theorem 5, for every natural number there exists a subset Q_k of power $\aleph_{k_0+k}$ of Q for which $\overline{\overline{Q}}_k = \overline{\overline{\Sigma}}_{Q_k} = \aleph_{k_0+k}$. Applying theorem 3 with $\mathfrak{q} = \aleph_{k_0+k}$ and $Q = Q_k$ we obtain that Q_k has a subset Γ_k with the property $T(\aleph_{k_0+k}, \mathfrak{p})$. Thus the conditions of the theorem 4 are satisfied with $\mathfrak{q} = \aleph_{\alpha+\omega}$, from where our assertion follows. The theorem is proved.

Theorem 7. *If* 1) $\mathfrak{n} = \mathfrak{p} = \aleph_0$ *and* $\mathfrak{q}$ *is a regular cardinal number, or if* 2) $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, $\mathfrak{q} = \aleph_{\alpha+2}$ *and* $\mathfrak{p} = \mathfrak{n} \leqq \aleph_\alpha$, *then* Q *has a subset with the property* $T(\mathfrak{q}, \mathfrak{n})$.

Proof. In the first case by the theorem 1 with $\mathfrak{n} = \aleph_0$, and in the second case by the theorems 1 and 2, Q has a subset Γ with the property $T(\mathfrak{q}, \mathfrak{r})$ where $\mathfrak{r} = \mathfrak{q}$ in the first case and $\mathfrak{r} = \aleph_{\alpha+1}$ in the second case. As $\overline{\overline{\Sigma}}_\Gamma = q$ and $\overline{\overline{\Pi}}_\Gamma < q$, we see that Γ has a subset Γ_1 of power $\mathfrak{q}$ such that, for every $x \in \Gamma_1$, we have $H_1(x) = H(x) - \Pi_\Gamma \neq 0$. Let $N(\Pi_\Gamma)$ be the set of all subsets of power less than $\mathfrak{n}$ of Π_Γ. Put $H_2(x) = H(x) \cap \Pi_\Gamma$. It is obvious that $\overline{\overline{N(\Pi_\Gamma)}} < \mathfrak{q}$ and $H_2(x) \in N(\Pi_\Gamma)$. It follows by the regularity of $\mathfrak{q}$ that Γ_1 has a subset Γ_2 of power $\mathfrak{q}$ such that for every $x, y \in \Gamma_2$ we have $H_2(x) = H_2(y)$. By the definition for every pair x, y of distinct elements of Γ we have $H_1(x) \cap H_1(y) = 0$. It follows that Γ_2 has the property $T(\mathfrak{q}, \mathfrak{n})$. The theorem is proved.

Theorem 8. *If* $\mathfrak{q}$ *is a singular cardinal number and if* 1) $\mathfrak{n} = \aleph_0$, *or if* 2) $\aleph_0 \leqq \mathfrak{n} < \mathfrak{q}$ *and* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ *for every* $\aleph_\alpha < \mathfrak{q}$, *then* Q *has a subset with the property* $T(\mathfrak{q}, [\mathfrak{n}\cdot\mathfrak{q}^*]^+)$.

Proof. Let $\{\eta_\xi\}_{\xi<\varphi_{\mathfrak{q}^*}}$ be a sequence of the type $\varphi_{\mathfrak{q}^*}$ of ordinal numbers such that $\eta_\nu < \eta_\mu$ $(\nu < \eta)$, $\max\{\mathfrak{q}^*, \mathfrak{n}\} < \aleph_{\eta_\xi+2} < \mathfrak{q}$ and $\mathfrak{q} = \sum_{\xi<\varphi_{\mathfrak{q}^*}} \aleph_{\eta_\xi+2}$. By theorem 5 for every ξ $(\xi < \varphi_{\mathfrak{q}^*})$ there exists a subset Q_ξ of Q such that

$$\overline{\overline{Q}}_\xi = \overline{\overline{\Sigma}}_{Q_\xi} = \aleph_{\eta_\xi+2}.$$

Applying the theorem 7 with $q = \aleph_{\eta_\xi+2}$ and $Q = Q_\xi$ we obtain that Q_ξ has a subset with the property $T(\aleph_{\eta_\xi+2}, \mathfrak{n})$. By theorem 4 the theorem is proved.

The following two theorems are due to P. ERDŐS.

Theorem 9. *If $\mathfrak{q}$ is a singular cardinal number and $\mathfrak{p} \leqq (\mathfrak{q}^*)^+$, then the answer to the problem, in general, is negative.*

Proof. Let

$$x_0, x_1, x_2, \ldots, x_\omega, x_{\omega+1}, \ldots, x_\xi, \ldots \qquad (\xi < \varphi_{\mathfrak{q}})$$

be any well-ordering of Q of the type $\varphi_{\mathfrak{q}}$. We define $H(x)$ as follows: Let $H(x_\xi) = 0$ if $\xi < \varphi_{\mathfrak{q}^*}$, and $H(x_\xi) = \{x_\eta, x_\xi\}$ if $\omega_\eta \leqq \xi < \omega_{\eta+1}$ $(0 < \eta < \varphi_{\mathfrak{q}^*})$. Let Γ be a subset of power $\mathfrak{q}$ of Q. It is obvious that $\overline{\overline{\Sigma}}_\Gamma = \mathfrak{q}$ and $\overline{\overline{\Pi}}_\Gamma = \mathfrak{q}^*$.

Theorem 10. *If $\mathfrak{q} = \aleph_{\alpha+1}$, $\mathfrak{s} = \aleph_\alpha$ is singular, $\mathfrak{p} \leqq \mathfrak{n}$, $\mathfrak{n} = (\mathfrak{s}^*)^+$ and $2^{\aleph_\beta} = \aleph_{\beta+1}$ for every β, then the answer to the problem, in general, is negative.*

Proof. Let

$$Q = R \cup (\bigcup_{\xi<\varphi_{\mathfrak{s}^*}} Q_\xi)$$

be a decomposition of Q into the sum of mutually disjoint non empty sets such that $\overline{\overline{R}} = \mathfrak{q}$ and $\overline{\overline{Q}}_\xi = \aleph_\alpha = \mathfrak{s}$, for any $\xi < \varphi_{\mathfrak{s}^*}$. Let P be the set of all functions $y(\xi)$ defined on the set of all ordinal numbers $\xi < \varphi_{\mathfrak{s}^*}$ and such that, for each $\xi < \varphi_{\mathfrak{s}^*}$, $y(\xi) \in Q_\xi$. It is clear that $\overline{\overline{P}} = \aleph_\alpha^{\mathfrak{s}^*}$. Thus, by the generalized continuum hypothesis, $\overline{\overline{P}} = \aleph_{\alpha+1}$. Let $\{x_\eta\}_{\eta<\varphi_{\mathfrak{q}}}$ and $\{y_\eta\}_{\eta<\varphi_{\mathfrak{q}}}$ be any wellordering of the type $\varphi_{\mathfrak{q}}$ of R and P, respectively. We define $H(x)$ as follows: Put $H(x) = 0$ if $x \in Q_\xi (\xi < \varphi_{\mathfrak{s}^*})$, and $H(x_\eta) = \{x_\eta\} \cup \{y_\eta(\xi)\}_{\xi<\varphi_{\mathfrak{s}^*}}$ for $x_\eta \in R$. Let $D = \{x_{\eta_\zeta}\}_{\zeta<\varphi_{\mathfrak{q}}}$ be a subset of type $\varphi_{\mathfrak{q}}$ of R. Let further D' be the set of elements $x_\tau \in D$ for which there exists at least one $\xi_0 < \varphi_{\mathfrak{s}^*}$ such that $y_\tau(\xi_0) \neq y_{\eta_\zeta}(\xi_0)$ for every $\eta_\zeta \neq \tau$. Clearly $\overline{\overline{D'}} \leqq \mathfrak{s}\,\mathfrak{s}^* = \mathfrak{s} = \aleph_\alpha$. Thus $\overline{\overline{D - D'}} = q = \aleph_{\alpha+1}$. By the definition of $H(x)$ it follows that $\overline{\overline{\Sigma}}_{D-D'} = \mathfrak{q}$. Put $\aleph_\alpha = \sum_{\nu<\varphi_{\mathfrak{s}^*}} \aleph_\nu$. There exists an increasing sequence $\{\xi_\nu\}_{\nu<\varphi_{\mathfrak{s}^*}}$ of the ordinal numbers $\xi < \varphi_{\mathfrak{s}^*}$ such that $\overline{\overline{\{y_{\eta_\zeta}(\xi_\nu)\}}}_{\zeta<\varphi_{\mathfrak{q}}} \geqq \aleph_\nu$ for every $\nu < y_{\mathfrak{s}^*}$. Suppose the contrary. Then there exists an ordinal number $\mu < \alpha$ so that $\overline{\overline{\{\varphi_{\eta_\zeta}(\xi)\}}}_{\zeta<\varphi_{\mathfrak{q}}} \leqq \aleph_\mu$ for every $\xi < \varphi_{\mathfrak{s}^*}$. But then $\overline{\overline{\{y_{\eta_\zeta}\}}}_{\zeta<\varphi_{\mathfrak{q}}} \leqq \aleph_\mu^{\mathfrak{s}^*} < \aleph_{\alpha+1}$, which is a contradiction. It follows by the definition of D and $H(x)$ that $\overline{\overline{\Pi}}_{D-D'} = \aleph_\alpha$. The theorem is proved.

Theorem 11. *Let $\mathfrak{r}$ be an arbitrary cardinal number, $\mathfrak{r} \geqq \aleph_0$, and $\aleph_\alpha$ a singular cardinal number such that α is not confinal to ω and $\aleph_\alpha > \mathfrak{r}$. If for every $\aleph_\beta$, $\aleph_\alpha > \aleph_\beta > \mathfrak{r}$, for which β is confinal to ω, the answer to the* *problem is affirmative with* $\mathfrak{q}=\mathfrak{p}=\aleph_\beta$ *and* $\mathfrak{n}=\mathfrak{r}$, *then there exists an ordinal number* $\beta_0<\alpha$ *such that the answer to the problem is affirmative with* $\mathfrak{q}=\aleph_\alpha$, $\mathfrak{p}=[\aleph_{\beta_0}\bar{\alpha}^*]^+$ *and* $\mathfrak{n}=\mathfrak{r}$.

Proof. Suppose that the conditions of the problem are satisfied with $\mathfrak{q}=\aleph_\alpha$ and $\mathfrak{n}=\mathfrak{r}$. According to the theorem 5, for every $\xi<\alpha$ for which $\max\{\mathfrak{n},\mathfrak{q}^*\}<\aleph_\xi$, there exists a subset Q_ξ of Q such that

$$\bar{\bar{Q}}_\xi=\bar{\bar{\Sigma}}_{Q_\xi}=\aleph_\xi .$$

Let D be the set of all ordinal numbers ξ which are confinal to ω and for which $\max\{\mathfrak{n},\mathfrak{q}^*\}<\aleph_\xi<\aleph_\alpha$. If $\xi\in D$, then by the condition Q_ξ has a subset Γ_ξ with the property $T(\aleph_\xi,\aleph_\xi)$ i. e.

$$\bar{\bar{\Sigma}}_{\Gamma_\xi}=\aleph_\xi \quad\text{and}\quad \bar{\bar{\Pi}}_{\Gamma_\beta}=\aleph_{f(\xi)}<\aleph_\xi .$$

We may assume that for every $x\in\Gamma_\xi$

$$H(x)-\Pi_{\Gamma_\xi}\neq 0.$$

Thus to every $\xi\in D$ there corresponds an ordinal number $f(\xi)$ such that $f(\xi)<\xi$. There exists an ordinal number β_0 and a sequence $\{\xi_\eta\}_{\eta<\alpha^*}$ of the type α^* of D such that $f(\xi_\eta)<\beta_0$ and $\lim\limits_{\eta<\alpha^*}\xi_\eta=\alpha$ (see [7]). It follows that for every Γ_{ξ_η} $(\eta<\alpha^*)$

$$\bar{\bar{\Pi}}_{\Gamma_{\xi_\eta}}<\aleph_{\beta_0}.$$

Consider now those $\xi_\eta(\eta<\alpha^*)$ for which η is of the form $\beta+1$. Let $\aleph_{\xi_\eta(\beta)}$ be a fixed regular cardinal number such that $\aleph_{\xi_\beta}<\aleph_{\xi_\eta(\beta)}<\aleph_{\xi_{\beta+1}}$. Let furthermore $\Gamma_{\xi_\eta(\beta)}$ be a subset of the power $\aleph_{\xi_\eta(\beta)}$ of Γ_{ξ_η}. It is obvious that $\Gamma_{\xi_\eta(\beta)}$ has the property $T(\aleph_{\xi_\eta(\beta)},\aleph_{\beta_0})$. As $\aleph_{\xi_\eta(\beta)}$ is regular and $\aleph_\alpha=\sum\limits_{\beta<\alpha^*}\aleph_{\xi_\eta(\beta)}$, therefore applying the theorem 4 with $\mathfrak{q}=\aleph_\alpha$, $\mathfrak{r}_0=\aleph_{\beta_0}$ and $Q_\beta=\Gamma_{\xi_\eta(\beta)}$, we obtain that $\bigcup\limits_{\beta<\alpha^*}\Gamma_{\xi_\eta(\beta)}$ has a subset with the property $T(\aleph_\alpha,[\aleph_\beta\bar{\alpha}^*]^+)$. The theorem is proved.

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