Note on the Theory of Monotone Operator Functions.

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The real-valued function $f(x)$ is said to be a *monotone operator function* in the interval (a, b) if, for any two bounded selfadjoint operators A, B on Hilbert space $\mathfrak{H}$, whose spectrum lies in (a, b), $A \geqq B$ implies $f(A) \geqq f(B)$. If we consider only operators on n-dimensional Euclidean space E_n, then these operators may be represented by matrices of type $n \times n$, and in this case a function $f(x)$ with the above property is called a *monotone matrix function* of order n.

The theory of monotone matrix functions has been developed by K. LÖWNER [4]; he gives first some necessary and sufficient conditions for a function to be a monotone matrix function of order n, and then, as a result of further deep investigations including questions of interpolation he arrives at the following criterion: A real-valued function $f(x)$ defined in (a, b) is monotone of arbitrarily high order n if and only if it satisfies the following condition (L): $f(x)$ is analytic in (a, b), can be analytically continued onto the entire upper half-plane, and has there a non-negative imaginary part.

The problem of monotone operator functions has recently been considered by J. BENDAT and S. SHERMAN [1][1]) Making use only of the necessity of LÖWNER's conditions for the monotonity of order n they proved that a function $f(x)$ with $f(0)=0$ is a monotone operator function in the interval $(-R, R)$ if and only if it is representable in the integral form

$$f(x)=\int_{-\frac{1}{R}}^{\frac{1}{R}} \frac{x}{1-tx}\, d\alpha(t) \tag{1}$$

with a non-decreasing bounded function $\alpha(t)$. (The restrictions $f(0)=0$ and $(-R, R)$ do not of course affect the generality; moreover, it is sufficient to consider only the case $R=1$.) They also proved that the class of monotone operator functions is identical with the class of monotone matrix functions of arbitrarily high order n and so it is characterized by LÖWNER's criterion.

[1]) The first results on this domain are due to HEINZ [2].

Now, an immediate proof of the equivalence of the conditions (L) and of the integral representation (1) would make possible to arrive to LÖWNER's criterion on the simpler way taken by BENDAT and SHERMAN. This equivalence can be proved by making use of a general theorem of R. NEVANLINNA [5] on asymptotic developments.[2]) However, the following direct proof may have, for its simplicity, some interest of its own.

Theorem. *Let the function $f(z)$ be defined, for $|z|<1$, by the convergent power series with real coefficients*

$$f(z)=\sum_{n=1}^{\infty} c_n z^n,$$

and suppose that $f(z)$ is analytic on the whole upper half-plane and that $\operatorname{Im} f(z) \geqq 0$ *for* $\operatorname{Im} z>0$. *Then $f(z)$ admits of the integral representation*

$$f(z)=\int_{-1}^{1} \frac{z}{1-tz}\, d\alpha(t) \tag{1}$$

with a non-decreasing, bounded function $\alpha(t)$.

Proof. Evidently, $f(z)$ can be continued analytically onto the lower half-plane too, thus the function $g(z)=-f\left(\frac{1}{z}\right)$ will be analytic on the entire complex plane, except possibly the interval $[-1,1]$ of the real axis. We have $\operatorname{Im} g(z) \geqq 0$ for $\operatorname{Im} z>0$, and, for $|z|>1$, we have the development

$$g(z)=-\sum_{n=1}^{\infty} \frac{c_n}{z^n}.$$

Choose a number $N>1$, and denote by K the circle of radius N with centre in the origin. We have

$$c_n=-\frac{1}{2\pi i}\int_K \zeta^{n-1} g(\zeta)\, d\xi \qquad (n=1.2, \ldots).$$

If we denote by Γ the part of the circle K in the upper half-plane, we have by $g(\bar{\zeta})=\overline{g(\zeta)}$

$$c_n=-\frac{1}{\pi}\operatorname{Im}\int_\Gamma \zeta^{n-1} g(\zeta)\, d\zeta.$$

Now consider the following oriented straight line segments in the complex plane: $A=[N, N+iy]$, $B=[N+iy, -N+iy]$, $C=[-N+iy, -N]$ $(y>0)$. We may write

$$c_n=-\frac{1}{\pi}\operatorname{Im}\left(\int_A+\int_B+\int_C\right) \zeta^{n-1} g(\zeta)\, d\zeta.$$

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[2]) See also [7], pp. 24—26.

Since $\zeta^{n-1}g(\zeta)$ is continuous at the points $\zeta=\pm N$, we have, for $y\to 0$,

$$c_n=-\frac{1}{\pi}\operatorname{Im}\int_B \zeta^{n-1}g(\zeta)d\zeta+O(y).$$

Now, if we make use of the binomial formula for $\zeta^{n-1}=(x+iy)^{n-1}$, and of the boundedness for $y\to 0$ of each of the integrals

$$\int_{-N}^{N} x^m \operatorname{Re} g(x+iy)dx,\quad \int_{-N}^{N} x^m \operatorname{Im} g(x+iy)dx \qquad (m=0,1,2,\ldots)^{3)}$$

we obtain that

$$(2)\qquad c_n=\frac{1}{\pi}\int_{-N}^{N} x^{n-1}\operatorname{Im} g(x+iy)dx+O(y).$$

With the help of the non-decreasing function

$$\alpha_y(t)=\frac{1}{\pi}\int_{-N}^{t}\operatorname{Im} g(x+iy)dx$$

this may be written in the form

$$c_n=\int_{-N}^{N} t^{n-1}d\alpha_y(t)+O(y).$$

The total variation $V(\alpha_y)$ of the function $\alpha_y(t)$ is

$$V(\alpha_y)=\alpha_y(N)-\alpha_y(-N)=\frac{1}{\pi}\int_{-N}^{N}\operatorname{Im} g(x+iy)dx,$$

that is, applying (2) with $n=1$,

$$(3)\qquad V(\alpha_y)=c_1+O(y).$$

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3) This follows from the boundedness, for $y\to 0$, of each of the integrals

$$J_m=\int_{-N}^{N} x^m g(x+iy)dy \qquad (m=0,1,\ldots).$$

To see this, first observe that the integrals

$$G_m=\int_{-N}^{N}(x+iy)^m g(x+iy)dx=\int_{-N+iy}^{N+iy} z^m g(z)dz \qquad m=0,1\ \ldots$$

are bounded as $y\to 0$ since the points $\pm N$ are in the domain of regularity of the functions $z^m g(z)$. Now, from the identity $x^m=(z-iy)^m$ it follows, using again the binomial formula,

$$J_m=\sum_{\nu=0}^{m}(-iy)^\nu\binom{m}{\nu}G_{m-\nu}=G_m-iy\sum_{\nu=1}^{m}(-iy)^{\nu-1}\binom{m}{\nu}G_{m-\nu}=G_m+O(y),$$

which proves our assertion.

So, making y converge to zero through a sequence y_n, we may apply the well-known theorem of HELLY. Thus there exists a non-decreasing function of bounded variation $\alpha(t)$ such that

$$c_n = \int_{-N}^{N} t^{n-1}\, d\alpha(t) \qquad (n=1,2,\ldots).$$

As N is an arbitrary number >1, we see that $\alpha(t)$ is constant outside $[-1, 1]$, so we have

$$c_n = \int_{-1}^{1} t^{n-1}\, d\alpha(t) \qquad (n=1,2,\ldots) \tag{4}$$

and, for $|z|<1$,

$$f(z) = \sum_{n=1}^{\infty} c_n z^n = \int_{-1}^{1} \sum_{n=1}^{\infty} t^{n-1} z^n\, d\alpha(t) = \int_{-1}^{1} \frac{z}{1-tz}\, d\alpha(t),$$

thus finishing the proof of the theorem.

Finally we add some remarks.

1. If we suppose that $\alpha(t)$ is conveniently normed, e. g. by demanding $\alpha(-1)=0$ and continuity from the left, $\alpha(t)$ is determined by (4) uniquely.

2. From the asymptotic equality (3) it follows that $V(\alpha)=c_1$, with $V(\alpha)$ the total variation of $\alpha(t)$.

3. The converse of the theorem is also true; every function of the form (1) has the properties enumerated in the theorem, as it can be seen by an elementary calculation.

4. Applying the substitution $\lambda = -\frac{M}{z}$ in the theorem we obtain for any function $g(\lambda)$ which is analytic everywhere except the real interval $[-M, M]$, tends to 0 for $\lambda \to \infty$, and has a non-negative imaginary part for λ in the upper half-plane, the integral representation

$$g(\lambda) = \int_{-M}^{M} \frac{d\beta(t)}{t-\lambda} \tag{5}$$

with a non-decreasing, bounded $\beta(t)$.

These conditions are fulfilled e. g. by the function $g(\lambda)=(R_\lambda u, u)$, if R_λ denotes the resolvent of a selfadjoint operator A, $M=\|A\|$, and u is an arbitrary element in Hilbert space.[4]) So we have a representation in the form (5) with $V(\beta)=\|u\|^2$, as a consequence of Remark 2. From these the spectral theorem for bounded selfadjoint operators follows by standard methods.[5])

[4]) See e. g. STONE [8].

[5]) This in essentially but a modern variant of the classical proof of E. HELLINGER. For the non-bounded case cf. LENGYEL [3] and NIEMINEN [6].

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