On the Jordan—Dedekind chain condition.

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Let S be a partly ordered set, $a, b \in S$, $a < b$. $R(a, b)$ (eventually with indices) denotes a chain in S with the least element a and the greatest element b. If $R(a, b)$ is finite and contains n elements, his length is $n-1$. A chain $R(a, b)$ is maximal, if it is not a proper subset of any chain $R_1(a, b)$ in S. Many important properties of the partly ordered set S can be proved under the assumption that (i) all bounded chains in S are finite and (ii) the following Jordan—Dedekind condition holds:

(JD) If $a, b \in S, a < b$, and $R_1(a, b), R_2(a, b)$ are maximal chains, then both these chains have the same length.

In a recent paper [1] G. SZÁSZ generalizes the concept of the length of a chain as follows: if a chain $R(a, b)$ is infinite, his length is the cardinal number of the set $R(a, b)$. The condition (JD) can now be considered in the generalized sense, without supposing that the lengths of $R_1(a, b), R_2(a, b)$ are finite.

It is well-known that every distributive lattice in which all bounded chains are finite satisfies the Jordan—Dedekind chain condition. In the paper [1] the interesting theorem (theorem 3) is stated:

There exists a distributive lattice which does not satisfy the (generalized) condition (JD)[1]).

In the present note we give a generalization of this theorem.

Let M be a non-empty set. Let us denote by $S(M)$ the set of all functions f defined on M such that, for every $i \in M$, $f(i)$ is a rational number, $f(i) \in [0, 1]$. $S(M)$ is partly ordered in the usual way: $f_1 \leqq f_2$ if and only if $f_1(i) \leqq f_2(i)$ for every $i \in M$. $S(M)$ is a distributive lattice; we shall denote the least and the greatest elements of $S(M)$ by f_0 and f_1, respectively.

Lemma 1. *If M is non-empty, then there exists in the lattice $S(M)$ a countable maximal chain $R_1(f_0, f_1)$.*

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1) The formulation of the example in the proof of this theorem is not correct; see the „Correction" on p. 270 of this volume.

Proof. Let R_1 be the set of all „constants" of $S(M)$ (i. e. of the $f \in S(M)$ with $f(i) = f(j)$ for all $i, j \in M$); the function $f \in R_1$, for which $f(i) = x$ ($x \in [0, 1]$, x rational) identically, we denote by f_x. Clearly, R_1 is a countable chain, containing f_0 and f_1. Let $g \in S(M), g \notin R_1$. Then there exist elements $i, j \in M$ for which $g(i) < g(j)$. We choose a rational number z such that $g(i) < z < g(j)$. Then we have $g(i) < f_z(i)$, $g(j) > f_z(j)$, and the elements g, f_z are incomparable. It follows that the chain R_1 is maximal. We shall say that R_1 is the *diagonal chain* in $S(M)$.

Lemma 2. *Let $M = [0, 1]$. In the lattice $S(M)$ there exists an uncountable maximal chain $R_2(f_0, f_1)$.*

Proof. For every $i \in M$ let R^i be the set of all $f \in S(M)$ with the property

$$j < i \Longrightarrow f(j) = 1, \quad j > i \Longrightarrow f(j) = 0.$$

Every R^i is a chain. It is clear that the set-theoretical sum $R_2 = \bigcup R^i$ is a chain containing f_0 and f_1. The chain R_2 is uncountable.

We will prove that the chain R_2 is maximal. Let the element $g \in S(M)$ be comparable with all $f \in R_2$, let $f_0 \neq g \neq f_1$. i) If there exists $i \in M$ such that $g(i) \in (0, 1)$, we choose $f_\alpha, f_\beta \in R^i$ such that $f_\alpha(i) < g(i) < f_\beta(i)$. Then it must be $f_\alpha(j) \leqq g(j) \leqq f_\beta(j)$ for every $j \in M$, thus $g \in R^i \subset R_2$. ii) Let us suppose that, for every $i \in M$, $g(i) = 0$ or $g(i) = 1$. If there exist $i, j \in M, i < j$ with $g(i) = 0$, $g(j) = 1$, then we consider the function $f^j \in R^j$ such that $f^j(j) = \frac{1}{2}$. The elements f^j, g are incomparable, contrary to the hypothesis. Hence if $g(i) = 0$, $g(j) = 1$, it must be $j < i$. Let M_1 be the set of all $i \in M$ such that $g(i) = 0$, $k = \inf M_1$. Clearly $g \in R^k \subset R_2$, and the chain R_2 is maximal. We shall say that R_2 is the *superficial chain* in $S(M)$.

The proof of the theorem 3 in [1] follows from lemma 1 and 2.

Remarks. 1) In the proof of lemma 2 the assumption $M = [0, 1]$ can be replaced by the following weaker one: M is an uncountable complete chain.

2) If we suppose only that M is an uncountable chain, the chain R_2 constructed in the proof of lemma 2 need not be maximal.

3) Let M be non-empty. Then the lattice $S(M)$ is not complete. We shall now suppose the axiom of choice and construct a complete distributive lattice which does not satisfy the condition (JD).

Let M be a non-empty set. We denote by $S^0(M)$ the lattice of all real functions f defined on M such that, for every $i \in M$, $f(i) \in [0, 1]$. Clearly the lattice $S^0(M)$ is isomorphic with the direct union ΠA_i ($i \in M$) where every A_i is isomorphic with the chain $A = [0, 1]$. The lattice A is complete and completely distributive (see [2], p. 146, (22'), and [3]), hence the lattice $S^0(M)$ is complete and completely distributive. The least (greatest) element of $S^0(M)$ will be denoted by f_0 (f_1).

Lemma 3. *Let M be non-empty. In the lattice $S^0(M)$ there exists a maximal chain $R_1(f_0, f_1)$ the length of which is c (i. e. the power of the continuum).*

See the proof of lemma 1.

Lemma 4. *Let $\alpha(M)$ be the cardinal number of the set M, let $\alpha(M) > c$. In the lattice $S^0(M)$ there exists a maximal chain $R_2(f_0, f_1)$, the length of which is $\alpha(M)$.*

Proof. Suppose the set M is well-ordered. We construct the maximal chain $R_2(f_0, f_1)$ as in lemma 2. The cardinal number of every chain $R^i, i \in M$, is c. The chains $R^i, R^j, i \neq j$ have not more than one element in common (the set $R^i \cap R^j$ contains one element if i covers j or i is covered by j in M). It follows that the cardinal number of the chain $R_2 = \bigcup R^i$ $(i \in M)$ is $c \cdot \alpha(M) = \alpha(M)$.

Lemma 5. *Let S be a lattice, $S = A \times B$, let $0_a, 1_a$ $(0_b, 1_b)$ be the least resp. the greatest element of A (B), let $R_1(0_a, 1_a)$, $R_2(0_b, 1_b)$ be a maximal chain in the lattice A resp. B. Then the set R of all elements of S which have the form*

$$\text{a)} \quad (a_i, 0_b), \quad a_i \in R_1(0_a, 1_a)$$

or

$$\text{b)} \quad (1_a, b_i), \quad b_i \in R_2(0_b, 1_b)$$

is a maximal chain in S with the least element $(0_a, 0_b)$ and with the greatest element $(1_a, 1_b)$.

Proof. Clearly, R is a chain in S containing the elements $(0_a, 0_b)$, $(1_a, 1_b)$. Suppose that the element $(a, b) \in S$ is comparable with all elements of the chain R. Then the element a resp. b is comparable with all elements of the chain $R_1(0_a, 1_a)$ resp. $R_2(0_b, 1_b)$. If $b = 0_b$, then clearly $(a, b) \in R$. If $b > 0_b$, then the element (a, b) is comparable with the element $(1_a, 0_b) \in R$; hence $a \geqq 1_a$, $a = 1_a$, consequently $(a, b) \in R$.

Lemma 6. *Let M_1, M_2 be non-empty, disjoint subsets of the set M with $M_1 \cup M_2 = M$. Then the lattice $S^0(M)$ is isomorphic with the direct union $S^0(M_1) \times S^0(M_2)$.*

The proof is clear.
Now we will prove the

Theorem. *Let α be a cardinal number, $\alpha \geqq c$. There exists a complete and completely distributive lattice S_α with the least element f_0 and the greatest element f_1, which has the following property: for any cardinal number β with $c \leqq \beta \leqq \alpha$, there exists in S_α a maximal chain $R_\beta(f_0, f_1)$ the length of which is β.*

Proof. Let M be a well-ordered set the cardinal number of which is α. We show that $S_\alpha = S^0(M)$ possesses the required property. For $\beta = c$ or $\beta = \alpha$ the statement is proved in the lemma 3 resp. 4. Let $c < \beta < \alpha$. Then there exists a subset $M_1 \subset M$ such that the cardinal number of M_1 is β. Denote $M_2 = M - M_1$, $A = S^0(M_1)$, $B = S^0(M_2)$. Let A_0 be the superficial chain in the lattice A (see lemma 4); let B_0 be the diagonal chain in the lattice B (see lemma 3). We denote the least and the greatest element in A (B) as in lemma 5. By lemma 5 the set of all elements of $A \times B$ which have the form a) or b) is a maximal chain R in $A \times B$. The length of the chain $A_0(B_0)$ is β (c), hence the length of the chain R is $\beta + c = \beta$. Thus by lemma 6 there exists a maximal chain $R_\beta(f_0, f_1)$ in the lattice $S^0(M)$ the length of which is β.

References.

[1] G. Szász, Generalization of a theorem of Birkhoff concerning maximal chains of a certain type of lattices, *these Acta*, **16** (1955), 89—91.

[2] G. Birkhoff, *Lattice theory* (American Math. Soc. Coll. Publ., vol. 25), revised edition, (New York, 1948).

[3] J. R. Büchi, Representation of complete lattices by sets, *Portugaliae Math.*, **11** (1952), 151—167.

(Received October 24, 1955.)