On a characterization of the classical orthogonal polynomials.

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1. We call classical orthogonal polynomial systems the Jacobi, Laguerre and Hermite polynomial systems, respectively all systems which can be derived from these by means of linear transformations of the variable. A common feature of all these orthogonal polynomial systems is that for each of them there exists a differential equation of the following form:

$$a(x)y''+b(x)y'=\lambda y \tag{1. 1}$$

which is satisfied by all polynomials $p_0, p_1, \ldots, p_n, \ldots$ of the system, and a corresponding sequence $\lambda_0(=0), \lambda_1, \ldots, \lambda_n, \ldots$ of values of the parameter λ.

Other orthogonal polynomial systems, satisfying in this sense a differential equation similar to (1. 1), are not known. In our present paper we shall prove that no such systems — beyond the three classical systems — exist.

The possibility to characterize the classical orthogonal polynomials as the only solutions of equation (1. 1) among orthogonal polynomials has been proposed by J. ACZÉL.[1])

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[1]) J. ACZÉL, Eine Bemerkung über die Charakterisierung der „klassischen" orthogonalen Polynome, *Acta Math. Hung.*, 4 (1953), 315—321. The problem can be formulated also as follows: Determine the solutions $\{y, \lambda\}$ of equation (1. 1), but instead of the usual boundary conditions put the condition that y be a polynomial. In which case can now a system of solutions $\{p_n, \lambda_n\}$ be found, such that $\{p_n\}$ is an orthogonal system with respect to a non-negative weight function?

A similar problem has been dealt with by the author in his paper „Über durch Sturm—Liouvillesche Differentialgleichungen charakterisierte orthogonale Polynomsysteme", *Publicationes Math. Debrecen,* 3 (1954), 297—304.

The first and second theorems of the present paper however express more and use different proofs as loc. cit.

2. Suppose that (1. 1) has the solutions $\{p_i, \lambda_i\}$ $(i=1, 2, \ldots)$ where p_i is a polynomial of i^{th} degree, with the main coefficient 1. Then we have, for an arbitrary solution $\{y, \lambda\}$,

$$\begin{vmatrix} 0 & 1 & -\lambda_1 p_1 \\ 2 & p_2' & -\lambda_2 p_2 \\ y'' & y' & -\lambda y \end{vmatrix} = 0. \tag{2. 1}$$

By comparing (2. 1) and (1. 1) it can be seen that $a(x)$ can be a polynomial of not higher than second degree, while $b(x)$ represents a polynomial of exactly first degree (if $\lambda_1 \neq 0$). Thus by means of a suitable linear transformation of the variable x the differential equation (1. 1) takes on the form

$$Qy'' + xy' = \lambda y \quad \text{with} \quad Q = ax^2 + bx + c. \tag{2. 2}$$

Here λ can be expressed as a function of n by substituting the polynomial p_n into equation (2. 2) and equating the coefficients of x^n on both sides. Thus we obtain

$$\lambda_n = n(n-1)a + n \quad \text{with} \quad n \geqq 2. \tag{2. 3}$$

It is known that the classical polynomial systems satisfy the following differential equations. Jacobi polynomials:

$$(x^2-1)y'' + [(\alpha+\beta+2)x + \alpha - \beta]y' = n(\alpha+\beta+n+1)y \tag{2. 4}$$
$$\alpha > -1, \ \beta > -1;$$

Laguerre polynomials:

$$-xy'' + (x-\alpha-1)y' = ny, \qquad \alpha > -1; \tag{2. 5}$$

Hermite polynomials:

$$-y'' + 2xy' = 2ny. \tag{2. 6}$$

Substituting $x = \dfrac{1}{\alpha+\beta+2}x' + \dfrac{\beta-\alpha}{\alpha+\beta+2}$ equation (2. 4) reduces to (2. 2), similarly the substitutions $x = x' + \alpha + 1$ and $x = \dfrac{x'}{2}$ reduce (2. 5) and (2. 6), respectively, to (2. 2). It can be directly seen that in the equation (2. 2) thus obtained we have

$$a \geqq 0 \quad \text{and} \quad c < 0. \tag{2. 7}$$

One verifies by a simple but longer reckoning that the only polynomial solutions of (2. 2) with (2. 7) are essentially the classical polynomials (see the cited paper of the author, pp. 300—301).

3. We prove that, among all orthogonal polynomial systems, the classical polynomial systems are the only ones which satisfy a differential equation of the type (1. 1).

Theorem 1. *Among the orthogonal polynomial systems only the three classical ones can satisfy an equation of type* (1. 1) *so, that each polynomial of the system represents a solution of the same differential equation for different* λ *values.*

We shall be able to prove even a *stronger* theorem[2]):

Theorem 2. *Let* $p_0, p_1, \ldots, p_n, \ldots$ *be polynomials of precisely* 0^{th}, *first*, $\ldots$, n^{th} *degree*, $\ldots$. *Further let the zeros of the system* $\{p_n\}$ *be distinct, real and separated. This last property means that, denoting the* k^{th} *zero of* p_n *by* α_{nk} *we have*

$$\alpha_{n+1,1} < \alpha_{n1} < \alpha_{n+1,2} < \alpha_{n2} < \cdots < \alpha_{nn} < \alpha_{n+1,n+1}. \tag{3. 1}$$

If each polynomial of the system $\{p_n\}$ *represents a solution of the differential equation* (1. 1) *with the corresponding parameter values* $\lambda_0, \lambda_1, \ldots, \lambda_n, \ldots$, *then the system* $\{p_n\}$ *cannot be but one of the three classical orthogonal systems.*

Proof. a) If $\lambda_1 \neq 0$ then, according to paragraph 2, the only polynomial solutions of the equation (2. 2) with (2. 7) are the classical polynomials. It can be directly seen that in (2. 2) $p_1 = x$ and $\lambda_1 = 1$.

b) By (2. 3) it can be stated that *the condition* $a \geqq 0$ *is equivalent to the condition* $\lambda_n > 0$ *(with* $n \geqq 2$*).*

By a) and b) it will be sufficient to prove that if condition (3. 1) holds for the zeros of $\{p_n\}$ then for the corresponding differential equation in its form (2. 2) we have

$$\lambda_1 = 1, \quad \lambda_n > 0 \quad (\text{for } n \geqq 2) \tag{A}$$

and

$$c < 0. \tag{B}$$

Proof of (A). Similarly to (2. 1) we can write for the solutions p_1, p_n and p_{n+1} the following equation:

$$D(x) = \begin{vmatrix} 0 & 1 & -\lambda_1 p_1 \\ p_n'' & p_n' & -\lambda_n p_n \\ p_{n+1}'' & p_{n+1}' & -\lambda_{n+1} p_{n+1} \end{vmatrix} = 0. \tag{3. 2}$$

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[2]) In fact, the zeros of a system of orthogonal polynomials having a non-negative weight function are distinct, real and separated.

With this expression we may derive relations between λ_n and the coefficients of p_n and p_{n+1}, respectively.

Introducing the substitution $x' = x + \alpha_{n+1,1}$ in $D(x)$, $D(\alpha_{n+1,1})$ will reduce to a very simple form, and the polynomials

$$p_n(x + \alpha_{n+1,1}) = p_n^*(x) = \sum_{k=0}^{n} a_{nk} x^k$$

will have only non-negative zeros in consequence of (3.1). We denote the roots of these polynomials again by

$$\alpha_{n+1,1} = 0 < \alpha_{n,1} < \alpha_{n+1,2} < \alpha_{n,2} < \cdots < \alpha_{nn} < \alpha_{n+1,n+1}$$

and have

$$D(\alpha_{n+1,1}) = \begin{vmatrix} 0 & 1 & -\lambda_1 a_{10} \\ 2a_{n2} & a_{n1} & -\lambda_n a_{n0} \\ 2a_{n+1,2} & a_{n+1,1} & 0 \end{vmatrix} = 0. \tag{3.3}$$

Here λ_n is identical to the λ_n figuring in (3.2), because λ_n is a function of a and n only, but the latter remain unchanged after the linear substitution mentioned. It can be seen from (3.3) that $\lambda_1 \neq 0$, for if $\lambda_1 = 0$, (3.3) would mean $\lambda_n a_{n0} a_{n+1,2} = 0$ i. e. $\lambda_n = 0$, the roots α_{nk} being positive. This is however impossible. Thus $\lambda_1 \neq 0$ and we can assume that $\lambda_1 = 1$.

Developing $D(\alpha_{n+1,1})$ and expressing (partly) its polynomial coefficients as elementary symmetric functions of the roots, we obtain

$$\frac{a_{n+1,2} a_{n0}}{a_{10}} \lambda_n = \sum_{\binom{n}{n-1}} \overbrace{\alpha_{n1} \ldots \alpha_{nn}}^{n-1} \cdot \sum_{\binom{n}{n-1}} \overbrace{\alpha_{n+1,2} \ldots \alpha_{n+1,n+1}}^{n-1} - \tag{3.4}$$
$$- \alpha_{n+1,2} \ldots \alpha_{n+1,n+1} \cdot \sum_{\binom{n}{n-2}} \overbrace{\alpha_{n1} \ldots \alpha_{nn}}^{n-2}.$$

Here the notation $a_{nk} = (-1)^{n+k} \sum_{\binom{n}{n-k}} \overbrace{\alpha_{n1} \ldots \alpha_{nn}}^{n-k}$ has been used.[3])

All α_{nk} values being positive, the sign of the coefficients a_{nk} will depend only on $n+k$ being even or odd. Thus the coefficient of λ_n must be positive in (3.4). It remains only to prove that the right side of (3.4) is also *positive*.

Let I be the first sum (of positive sign) and II be the second (of

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[3]) More explicitly $\sum_{\binom{n}{n-k}} \overbrace{\alpha_{n1} \ldots \alpha_{nn}}^{n-k}$ will mean that, taking $n-k$ different elements of the set $\{\alpha_{n1}, \ldots, \alpha_{nn}\}$, they are multiplied, and the possible $\binom{n}{n-k}$ products are summed.

negative sign). We shall establish a correspondence between II and I in the following way: Delete from the set

(3. 5) $$\alpha_{n1}, \ldots, \alpha_{nn}, \alpha_{n+1,2}, \ldots, \alpha_{n+1,n+1}$$

the elements α_{nk} and μ_{ni} $(k < i)$ and multiply the remainder with each other. We thus obtain a term of II (defined by the indices i, k). To this term we let correspond the term of the sum I which can be obtained by deleting the elements α_{nk} and $\alpha_{n+1,i}$ of the set (3. 5) and multiplying the remaining elements. Thus for every term of sum II a corresponding term of I can be found. Every term of the sum I having a greater absolute value then the corresponding term of II, the right hand side of (3. 4) must be positive.

Thus our statement concerning $\lambda_n > 0$ (with $n \geqq 2$) has been proved.

Proof of (B). As in equation (2. 2) the zeros of $p_1 = x$ and of the solution p_2 are separated (in the sense already explained), thus $p_2(0) < 0$.

On the other side, we obtain by (2. 1) (taking $p_1 = x$)

$$\lambda_2 p_2 - \lambda_1 x p_2' = Q = ax^2 + bx + c$$

and thus

$$\lambda_2 p_2(0) = c.$$

Since $\lambda_2 > 0$, (B) must also hold.

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