On a useful lemma for abelian groups.

By L. FUCHS in Budapest.

I. This note is mainly of methodical character. It has for its origin the observation that, in the theory of abelian groups, if one has to prove the direct summand property of a given subgroup H, then the arguments are the same in all cases: one starts with the zero subgroup or with some appropriate subgroup of the given group[1]) G, extends it to a subgroup M maximal with respect to the property of disjointness from H, and then tries to prove that the assumption that $H+M$ [2]) is a proper subgroup of G leads to a contradiction. In order to avoid superfluous repetitions, it is natural to try to get a lemma from which the results of the mentioned kind may be obtained without repeating the common inferences.

An observation of this type is due to I. KAPLANSKY; in his booklet [2, p. 8] he has called the attention to the fact that — using the same notations as above — $G/(H+M)$ is necessarily a torsion group. Our lemma is a far-reaching generalization of this remark: it not only tells us that $G/(H+M)$ is a torsion group, but it establishes an isomorphism between the p-layer[3]) of $G/(H+M)$ and some subgroup of H/pH. The application of our lemma then reduces to the proof that this subgroup of H/pH collapses to 0. We shall also see that our lemma may successfully be applied to verifying certain properties of the factor group $G/(H+M)$ also when H is no direct summand, and to solve a problem concerning direct summands in a certain stricter sense than usual.

II. Now let us formulate:

L e m m a . *Let G be an abelian group, H a subgroup of G and M an arbitrary subgroup of G with the properties: $M\cap H=0$; $M\subset N$ implies $N\cap H\neq 0$. Then the group $G^*=H+M$ satisfies:*

---

[1]) Since we shall deal throughout with abelian groups only, we may write "group" for the longer term "abelian group".

[2]) The sign $+$ denotes (besides group operation) direct sum.

[3]) The p-layer of a group G, in sign $G[p]$, consists of all elements x of G for which $px=0$. (p denotes always a prime number.)

(a) G/G^* *is a torsion group;*

(b) $(G/G^*)[p] \cong (\{pG, M\} \cap H)/pH$.

In particular, (b) implies that $(G/G^*)[p]$ is isomorphic to some subgroup of H/pH.

Proof. (a) is almost evident (see [1]): $x \in G, x \notin G^*$ implies $\{x, M\} \cap H \neq 0$, i. e. we have $h = nx + m \neq 0$ ($h \in H, m \in M$, n integer), $nx = h - m \in G^*$. The hypothesis $M \cap H = 0$ guarantees that $n \neq 0$.

To prove (b), observe that all the elements $\neq 0$ of both groups in consideration are of order p. Now let[4]) $\bar{x} \in (G/G^*)[p]$, then $px = h + m$ ($h \in H, m \in M$) and clearly $h = -px + m \in \{pG, M\} \cap H = H_1$. Consider the correspondence $x \to h$. If $\bar{x}_1 = \bar{x}_2$, then $x_1 \to h_1$ and $x_2 \to h_2$ imply $h_1 \equiv h_2$ (mod pH); in fact, from $px_1 = h_1 + m_1, px_2 = h_2 + m_2$ and $x_1 - x_2 = h_3 + m_3$ ($h_3 \in H, m_3 \in M$) we conclude $ph_3 + pm_3 = (h_1 - h_2) + (m_1 - m_2)$, that is, $h_1 - h_2 = ph_3 \in pH$. On the other hand, under $x \to h$ all of H_1/pH is exhausted, for if $h \in \{pG, M\} \cap H, h \notin pH$, then $h = py - m$ ($y \in G, m \in M$), and here $y \notin G^*$, because $y = h' + m'$ ($h' \in H, m' \in M$) would imply $py = ph' + pm' = h + m, h = ph' \in pH$; thus $y \to h$. Further, if $x_1 \to h_1$, $x_2 \to h_2$ and $h_1 \equiv h_2$ (mod pH), then $\bar{x}_1 = \bar{x}_2$. For putting $px_i = h_i + m_i$ ($h_i \in H, m_i \in M$), $h_1 - h_2 = ph$ ($h \in H$), in case $x_1 - x_2 - h \in M$ we get $x_1 - x_2 \in H + M = G^*$ and are therefore ready. In case $x_1 - x_2 - h \notin M$ there is a nonzero element z in $\{M, x_1 - x_2 - h\} \cap H$, $z = m + k(x_1 - x_2 - h)$ ($m \in M$) where on account of $p(x_1 - x_2 - h) = m_1 - m_2 \in M$ we must have $(k, p) = 1$ (otherwise we should obtain $z \in M \cap H = 0$). Now from $k(x_1 - x_2 - h) \in H + M$ and $p(x_1 - x_2 - h) \in M$ we are led to $x_1 - x_2 - h \in H + M$, $x_1 - x_2 \in G^*$. The correspondence $x \to h$ between $(G/G^*)[p]$ and H_1/pH is therefore one-to-one. Since it carries sums into sums, we arrive at the desired isomorphism.

III. Now let us consider some applications of the Lemma.

1. If H is a complete group, i.e. $pH = H$ for every prime p, then Lemma tells us that every p-layer of G/G^* vanishes, i. e. $G^* = G$. Consequently, *a complete group is a direct summand of every containing group*[5])

2. For any H we get that the p-component of the factor group G/G^* is at most of power[6]) $|H/pH| \cdot \aleph_0$, thus

$$|G/G^*| \leqq \aleph_0 \cdot \sum_p |H/pH|.$$

---

[4]) The bar will indicate cosets modulo G^*.

[5]) This is a well-known result, see Baer [1], Kurosh [4] or Kaplansky [2].

[6]) For a set S, we denote by $|S|$ the power of S.

Hence it also follows:

$$|G/G^*| \leqq \aleph_0 \cdot |H|.$$

(The main interest of these inequalities lies in the fact that they are for every G containing H.)

3. Let H be a cyclic p-group, $H \cong \mathfrak{C}(p^n)$. Then Lemma implies that G/G^* is a p-group which either vanishes or has the layer $\mathfrak{C}(p)$. Therefore $G/G^* \cong \mathfrak{C}(p^k)$ with $0 \leqq k \leqq \infty$.[7])

4. If H is an infinite cyclic group, then G/G^* is isomorphic to some subgroup of the group $\mathfrak{C}$ of all finite rotations of the circle.

5. Let G be a serving subgroup of the additive group of p-adic numbers and H a serving subgroup of G. Then $H/pH \cong \mathfrak{C}(p)$ and $qH = H$ for all primes $q \neq p$. Since G is known to be directly indecomposable,[8]) we have $G/G^* \cong \mathfrak{C}(p)$.

6. Next let G be the complete direct sum of the cyclic groups of prime order $\mathfrak{C}(2), \mathfrak{C}(3), \ldots, \mathfrak{C}(p), \ldots$ and H the discrete direct sum of the same groups. Then H is the torsion subgroup of G; the factor group G/H is complete and therefore H is not a direct summand of G. Now G/G^* is again complete (as a homomorphic image of G/H) and — on using Lemma — it is easy to see that $G/G^* \cong \mathfrak{C}$, the group of all finite rotations of the circle.

7. Let G be a p-group with elements of bounded order $\leqq p^n$, let a denote an element of order p^n and $H = \{a\}$. We have $p^{n-1}H_1 \subseteq \{p^n G, p^{n-1}M\} \cap p^{n-1}H = p^{n-1}M \cap p^{n-1}H = 0$, consequently $H_1 \subseteq \{pa\} = pH$. This implies that $\{a\}$ *is a direct summand of* G.

8. We proceed to consider the case when G is arbitrary, and H is a serving subgroup of G and is the direct sum of cyclic groups of the same order p^n. Since, by servingness, $p^n G \cap H = p^n H = 0$, we may choose M subject to the restriction $p^n G \subseteq M$. Then we have:

$$p^{n-1}H_1 \subseteq \{p^n G, p^{n-1}M\} \cap p^{n-1}H \subseteq M \cap p^{n-1}H = 0,$$

and this implies, owing to $H \cong \Sigma\, \mathfrak{C}(p^n)$, that $H_1 \subseteq pH$. In conclusion, H is a direct summand of G. Hence it follows KULIKOV's result [3]: *a serving subgroup of bounded order is always a direct summand.*

Let us observe that the same argument establishes also the main result of SZELE's paper [5].

9. Let G be a p-group, and H a serving subgroup of G such that H contains $(p^n G)[p]$, but no element of $G[p]$ not in $(p^{n-1}G)[p]$, i. e. from the

[7]) $\mathfrak{C}(p^\infty)$ is PRÜFER's group of type p^∞.

[8]) See e. g. KUROSH [4].

layer of G it contains each element of height $\geqq n$, but no element of height $\leqq n-2$. If $H+M\subset G$, then take an $h=pg+m(g\in G, m\in M)$ in H but not in pH, of a possible least order p^s. Then $p^s g+p^{s-1}m=$ $=p^{s-1}h\neq 0$ implies $p^{s-1}m\neq 0$, for in the contrary case, by the servingness of H, we should have $p^{s-1}h=p^s h'(h'\in H)$ and $h-ph'$ would be an adequate element of a less order than h. Thus the height of $p^{s-1}h$ in G is exactly $s-1$, whence $s\geqq n$. Further, obviously, $p^s g$ is of order p (for $p^s h=p^s m=0$), and hence from $s\geqq n$ and the hypothesis on H it results $p^s g\in H$. But then $p^{s-1}h=p^s g+p^{s-1}m$ implies $p^{s-1}m=0$, a contradiction. Therefore, H is a direct summand of G. — Of course, a similar result holds for torsion groups.

10. We turn our attention to the following problem.

Let G be an abelian group; *find all subgroups H of G which are direct sumands in the strict sense that for every maximal M with $M\cap H=0$ one has $G=H+M$.* — We have seen that the complete subgroups H are always direct summands in this strict sense too. The same is true for the group H in 9.

Suppose $G=H+K$ and M is maximal with respect to the property of disjointness from H. If $H+M\subset G$, then there exists a prime p such that $H_1=\{pG, M\}\cap H\supset pH$, i. e. there is an $h=pk+m\in H(k\in K, m\in M)$ with $h\notin pH$.

Assume k is of finite order $p^t s$ with $(p,s)=1$; then we have again $sh=p(sk)+sm\in H$, $sh\notin pH$, consequently, we may suppose h so to be chosen that the order of k is a prime power p^t. Now $t\geqq 2$, for $h\in M$ is impossible. By multiplication by p^{t-1} we get $p^{t-1}h=p^t k+p^{t-1}m=p^{t-1}m$, whence $p^{t-1}h=0$ and so[9] $O(h)<O(k)$. — Conversely, if H and K contain elements h and k, respectively, such that $O(h)<O(k)=p^t$ and $h\notin pH$, then H is no direct summand in the strict sense. In fact, we first show that $\{h-pk\}\cap H=0$ holds. For $p^r(h-pk)\in H$ implies $p^{r+1}k\in H$, $p^{r+1}k=0$, $t\leqq r+1$, $p^r h=0$ and $p^r(h-pk)=0$. Therefore, we can choose an M with $h-pk\in M$. But then $h=pk+(h-pk)\in\{pG, M\}\cap H$, $h\notin pH$ and Lemma implies $H+M\subset G$.

Next assume k is of infinite order. Then there is no need to get further properties of k, considering that, conversely, in case K contains an element k of infinite order, then for every $h\in H$, $h\notin pH$ one has $\{h-pk\}\cap H=0$, and again $H+M\subset G$ provided M is chosen to contain $h-pk$.

---

[9]) By $O(x)$ we denote the order of the group element x.

Consequently, *a direct summand* H *of* G, $G = H + K$, *satisfies* $G = H + M$ *for every* M *maximal with respect to the property* $M \cap H = 0$, *if and only if either*

1. H *is a complete group, or*

2. K *is a torsion group the elements of whose p-components are of order not exceeding the order of any* $h \in H$ *with* $h \notin pH$.

This result states that examples 1 and 9 essentially exhaust all cases in which H is a direct summand in the stated stricter sense.

Bibliography.

[1] R. Baer, The subgroup of elements of finite order of an abelian group. *Annals of Math.* (2), **37** (1936), 766—781.

[2] I. Kaplansky, *Infinite abelian groups,* University of Michigan Publ. in Math., Nr 2 (Ann Arbor, 1954).

[3] Л. Я. Куликов, К теории абелевых групп произвольной мощности, Мат. Сборник, **9** (51) (1941), 165—181.

[4] А. Г. Курош, Теория групп (Москва, 1953).

[5] T. Szele, On direct decompositions of abelian groups. *Journ. London Math. Soc.*, **28** (1953), 247—250.

(*Received April 7, 1956.*)