On quasi nil groups.

By L. FUCHS in Budapest.

§ 1. Introduction.

In his paper [6][1]) T. SZELE has called an additive abelian[2]) group G a nil group, if there exists up to isomorphism only one ring R whose additive group is isomorphic to G, namely the zero ring in which any two elements have 0 as product. He has shown that the torsion nil groups coincide with the torsion divisible[3]) groups and that there do not exist mixed nil groups, while the problem of characterizing by group invariants the torsion free nil groups remained open. In an other paper [7] he investigated those groups G over which exactly two non-isomorphic rings may be defined[4]) (he called them quasi nil groups of species 2); these results are almost complete in the sense that the problem is reduced to that of torsion free nil groups.

Our present aim is to characterize the *quasi nil groups* (of finite species)[5]), i.e. those abelian groups G over which but a finite number of non-isomorphic rings can be defined. We shall discuss the case of torsion, torsion free and mixed groups separately. It will turn out that the main difficulty lies again in the torsion free case where our results are again far from giving an explicit description of the structure of the groups in question.

Our main results are contained in Theorems 1—3.

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1) The numbers in square brackets refer to the Bibliography given at the end of this note.

2) We shall throughout consider abelian groups, therefore henceforth „group" is used for the longer phrase „abelian group" (with additive notation).

3) For the terminology and basic facts on abelian groups we refer to KUROSH [5] or KAPLANSKY [3].

4) We say the ring R is defined over the group G if the additive group of R is isomorphic to G.

5) There is a simple difference between the terminology used by SZELE and that used here: he meant by a quasi nil group a quasi nil group of species 2, while we mean thereby one of finite species.

§ 2. The torsion case.

We begin with the following two lemmas which are essential in the proof.

Lemma 1. *If G is a torsion group which is not divisible, or is a mixed group whose torsion subgroup is not divisible, then G has a cyclic direct summand $\mathcal{C}(p^k)$ of order p^k, k a natural integer*[6]).

For the proof we refer to KULIKOV [4] or SZELE [8].

Lemma 2. *In a p-ring R the elements of infinite height annihilate every element of the ring.*

See e. g. SZELE [6] or FUCHS [2].

Now let G be a torsion quasi nil group. G can have but a finite number of p-components G_p which are not divisible. In fact, in the contrary case, in view of Lemma 1, an infinity of G_p would be decomposable as $G_p = \mathcal{C}(p^k) + G_p'$ and we may define over $\mathcal{C}(p^k)$ a ring $I(p^k)$ [the residue class ring of the rational integers modulo p^k], while over G_p' and over all other G_q $(q \neq p)$ zero rings, and then form their direct sum in order to obtain pairwise non-isomorphic rings over G. By Lemma 2, the divisible p-components of G are zero rings and it is clear that the non-divisible ones must again be quasi nil groups.

Next suppose G_p is a quasi nil p-group and let B_p be a basic subgroup of G_p. We shall show that B_p is finite. For, in the contrary case let $a_1, a_2, \ldots$ be a countable set of basis elements of cyclic subgroups in a direct decomposition of B_p. Each

$$B_p^{(n)} = \{a_1\} + \cdots + \{a_n\} \qquad (n = 1, 2, \ldots)$$

is a direct summand of G_p, $G_p = B_p^{(n)} + G_p^{(n)}$, and if we define over $G_p^{(n)}$ the zero ring, over each $\{a_i\}$ $(i = 1, \ldots, n)$ a ring $I(p^{n_i})$ where p^{n_i} is the order of a_i, then we obtain a ring R_n for each n. It is obvious that these rings R_n are not isomorphic for different integers n, because the orders of $B_p^{(n)}$ — $B_p^{(n)}$ may be defined as a complementary direct summand of the annihilator $G_p^{(n)}$ of G_p— are different.

Considering that B_p is thus finite, it follows that it is a direct summand of G_p,

$$G_p = B_p + D_p$$

where D_p is a divisible group. Consequently, a torsion quasi nil group G has the form

$$G = B + D \qquad (B \text{ finite, } D \text{ divisible}). \tag{1}$$

[6]) We denote by $\mathcal{C}(n)$ the cyclic group of order n, by $\mathcal{C}(p^\infty)$ the group of type p^∞ and by $\mathfrak{R}$ the additive group of the rationals.

Conversely, assume that G is a torsion group of the form (1) and R is a ring with G as additive group. In R, the p-components belonging to different primes annihilate one another, hence Lemma 2 implies that the elements of D are annihilators of the whole ring R. B as a finite group has the form $B=\{a_1\}+\cdots+\{a_t\}$ where a_i are of prime power orders. Consider the group A generated by B and by all products $a_i a_j$ $(i, j=1, \ldots, t)$. If $a_i a_j$ lies outside B, then its D-component in (1) is an annihilator, so that the *subring* generated by B must coincide with A. Since A is again finite, we conclude that there is a divisible subgroup D_1 of finite rank r in D such that $A \subseteq B+D_1$. Each $a_i a_j$ increases the rank at most one, thus we have $r \leqq t^2$. Further, $mB=0$ implies $mA=0$, i. e. A belongs to[7] $B+D_1[m]=A_1$. It results that all the products of the elements of R belong to a finite subgroup of G which may be chosen — up to automorphism — independently of the product definition of R. Since there is but a finite number of possibilities for defining a ring over a finite group, we arrive at

Theorem 1. *A torsion group G is a quasi nil group if and only if it is a direct sum of a finite group and a divisible group.*

§ 3. The torsion free case.

Let G be a torsion free quasi nil group and R a ring, different from the zero ring, over G. We may alter the multiplication ab of the elements a, b of R by setting $a \times_n b = nab$ for some fixed natural integer n. We then get rings R_n $(n=1, 2, \ldots)$ with the same additive group G. No R_n is a zero ring and by hypothesis among the R_n there exists but a finite number of non-isomorphic rings; let these be $R_{m_1}, R_{m_2}, \ldots, R_{m_t}$. Thus, for each n, R_n is isomorphic to some $R_{m_j} (j=1, \ldots, t)$.

Next take into account that, by definition, all the products in R_n belong to nG, i. e. $R_n^2 \subseteq nG$. If $R_{r_1}, R_{r_2}, \ldots$ are isomorphic to R_{m_1}, then in R_{m_1} all the products $a \times_{m_1} b = m_1 ab$ belong to $\bigcap_i r_i G$. Thus if $m=m_1 \ldots m_t$, then for every pair of elements a, b we have $mab \in \bigcap_n nG$ where n ranges over all natural integers. (Note that R_n is isomorphic to a certain R_{m_j}!) Therefore mab, and hence ab is divisible by every integer n, i. e., in R every product belongs to the maximal divisible subgroup D of G. $D \neq 0$, for G is not a nil group.

By a known result, D is a direct summand of G, $G=D+H$ where H contains no nonzero divisible subgroup (i. e. it is reduced), further D is the

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[7]) For a group G, $G[m]$ denotes the set of all $x \in G$ with $mx=0$.

direct sum of groups $\mathfrak{R}$ isomorphic to the additive group of the rationals, $D=\Sigma\mathfrak{R}$. Here the number of direct summands cannot exceed 1, for every algebraic number field of degree 2 over the rationals has an additive group of type $\mathfrak{R}+\mathfrak{R}$, and there is an infinity of non-isomorphic such fields. Thus G is of the form $G=\mathfrak{R}+H$ where the reduced group H must be a nil group, for otherwise we could define over G a ring in which not all the products belong to $\mathfrak{R}$.

The group H must be of finite rank. For, assume H is of infinite rank and let $[b_1,\ldots,b_\alpha,\ldots]$ be a maximal independent system in H and $b_1,\ldots,b_n,\ldots$ a countable (proper or improper) subsequence of it. For each n we define a ring R_n by putting 1. $b_\alpha b_\beta=0$ if α and β are different, 2. $b_\alpha^2=0$ or $=b_0$ according as $\alpha=0,1,2,\ldots,n-1$ or α is different from these indices. Here b_0 denotes an arbitrary nonzero element of $\mathfrak{R}$. Knowing the products of the b_α, the distributive law enables us to extend the multiplication to the whole of G (all the products belong to $\mathfrak{R}$!). Since any product of more than two factors vanishes, the associative law holds, and we conclude that R_n is indeed a ring. In R_n, any element of the form $\lambda_0 b_0+\lambda_1 b_1+\cdots+\lambda_{n-1}b_{n-1}$ (λ_i rational) is an annihilator of R_n, while any element containing a summand $\lambda_\alpha b_\alpha$ with $\lambda_\alpha\neq 0$ and $\alpha\neq 0,1,\ldots,n-1$, is no annihilator, for it does not vanish multiplying it by b_α. Thus, the rank of the annihilator ideal of R_n is just n, consequently, $n\neq m$ implies that R_n and R_m are not isomorphic and thus H is necessarily of finite rank.

If $H=0$, then $G=\mathfrak{R}$ and there are two non-isomorphic rings over $\mathfrak{R}$, namely the rational number field and a zero ring.

If $H\neq 0$, let the rank of H be the natural integer r. We denote by b_0 a nonzero element of $\mathfrak{R}$, and by $[b_1,\ldots,b_r]$ a maximal independent system of H. Our aim is to get information on all rings over $\mathfrak{R}+H$. For this purpose it is sufficient to know all products $b_i b_j$. Since they belong to $\mathfrak{R}$, we set

(2) $$b_i b_j=\lambda_{ij}b_0 \qquad (\lambda_{ij} \text{ rational})$$

for $i,j=0,\ldots,r$. The λ_{ij} may arbitrarily be chosen, only the associative law $(b_i b_j)b_k=b_i(b_j b_k)$ must be fulfilled. This is equivalent to

(3) $$\lambda_{ij}\lambda_{0k}=\lambda_{jk}\lambda_{i0} \qquad (i,j,k \text{ arbitrary}),$$

and therefore we assume (3) to hold. Now we distinguish two cases according as $\lambda_{00}\neq 0$ or $=0$.

Case 1. $\lambda_{00}\neq 0$. There is no loss of generality in assuming $\lambda_{00}=1$, since this can be achieved by an eventual alteration of the choice of b_0 in $\mathfrak{R}$.[8])

[8]) It suffices to replace b_0 by $\lambda_{00}^{-1}b_0$.

Under this assumption, (3) implies for the case $k=0$

(4) $$\lambda_{ij}=\lambda_{j0}\lambda_{i0}$$

whence it follows that $\lambda_{ij}=\lambda_{ji}$, i. e. the ring is necessarily commutative. Write $\lambda_{i0}=\lambda_{0i}=\lambda_i$, then (2) becomes $b_ib_j=\lambda_i\lambda_jb_0$ ($\lambda_0=1$) and (3) is automatically satisfied. — Let another ring be defined over G with the rule $b_i\times b_j=\mu_i\mu_jb_0$ ($\mu_0=1$) where the μ_i are arbitrary rationals. Define a (group) automorphism α of G by putting

$$b_i^\alpha=b_i+(\mu_i-\lambda_i)\,b_0 \qquad (i=0,1,\ldots,r).$$

It is obvious that α induces in fact an automorphism of G. Take into account that

$$b_i^\alpha b_j^\alpha=[b_i+(\mu_i-\lambda_i)\,b_0]\,[b_j+(\mu_j-\lambda_j)\,b_0]=(\mu_ib_0)(\mu_jb_0)=\mu_i\mu_j\,b_0=b_i\times b_j$$

(note that b_i behaves like λ_ib_0 under multiplication), and then conclude that under α, the rings defined by the λ_i and the μ_i, respectively, are isomorphic. Thus all rings defined over G with $\lambda_{00}\neq0$ are isomorphic.

Case 2. $\lambda_{00}=0$. Then from (3) in case $k=0$, $i=j$ we obtain $\lambda_{i0}^2=0, \lambda_{i0}=0$, and similarly, $\lambda_{0i}=0$, that is, $\mathfrak{R}$ is an annihilator of the ring. (3) shows that $\lambda_{ij}\,(i,j=1,\ldots,r)$ are not subject to any condition. Each ring R over G thus defines, in view of (2), a square matrix

$$A=\begin{pmatrix}\lambda_{11}\,\lambda_{12}\ldots\lambda_{1r}\\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\\ \lambda_{r1}\,\lambda_{r2}\ldots\lambda_{rr}\end{pmatrix}$$

with arbitrary rational elements. Another ring S over G gives rise to a matrix

$$M=\begin{pmatrix}\mu_{11}\,\mu_{12}\ldots\mu_{1r}\\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\\ \mu_{r1}\,\mu_{r2}\ldots\mu_{rr}\end{pmatrix}$$

relative to the same independent set $b_0, b_1, \ldots, b_r$. Let α be a (group) automorphism of G with

$$b_0^\alpha=\varrho_0b_0, \qquad b_i^\alpha=\sum_{k=0}^{r}\varrho_{ik}b_k \qquad (i=1,\ldots,r)$$

where ϱ_0, ϱ_{ik} are certain rational numbers. Before passing on we remark that α induces an automorphism α^* of H by setting $b_i^{\alpha^*}=\sum_{k=1}^{r}\varrho_{ik}b_k \quad (i=1,\ldots,r)$, the matrix of α^* is

$$P=\begin{pmatrix}\varrho_{11}\,\varrho_{12}\ldots\varrho_{1r}\\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\;\cdot\\ \varrho_{r1}\,\varrho_{r2}\ldots\varrho_{rr}\end{pmatrix}$$

and any automorphism α^* of H may be extended (in several ways) to automorphisms α of G, by choosing arbitrary rationals $\varrho_0, \varrho_{10}, \dots, \varrho_{r0}$. The two rings R and S defined over G are isomorphic if and only if there is an automorphism α of G such that the elements b_i^α may be multiplied in R in the same way as the elements b_i in S, i. e.

$$b_i^\alpha b_j^\alpha = \left[\sum_{k=0}^{r} \varrho_{ik} b_k\right]\left[\sum_{l=0}^{r} \varrho_{jl} b_l\right] = \sum_{k=1}^{r}\sum_{l=1}^{r} \varrho_{ik}\varrho_{jl}\lambda_{kl} b_0$$

is equal to $\mu_{ij}(\varrho_0 b_0)$ for $i, j = 1, \dots, r$. The condition obtained may be written in the matrix form

$$\begin{pmatrix} \varrho_{11} \dots \varrho_{1r} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \varrho_{r1} \dots \varrho_{rr} \end{pmatrix} \begin{pmatrix} \lambda_{11} \dots \lambda_{1r} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \lambda_{r1} \dots \lambda_{rr} \end{pmatrix} \begin{pmatrix} \varrho_{11} \dots \varrho_{r1} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \varrho_{1r} \dots \varrho_{rr} \end{pmatrix} = \varrho_0 \begin{pmatrix} \mu_{11} \dots \mu_{1r} \\ \cdot\;\cdot\;\cdot\;\cdot\;\cdot \\ \mu_{r1} \dots \mu_{rr} \end{pmatrix},$$

that is,

$$P \Lambda P' = \varrho_0 M \tag{5}$$

where P' denotes the transpose of P. Calling two matrices Λ and M *H-equivalent* if there is an automorphism α^* of H with the matrix P and there is a rational number ϱ_0 such that (5) holds, we get an equivalence relation among the $r \times r$ square matrices with rational elements. Our arguments above show that two rings over G are isomorphic if and only if the corresponding matrices Λ and M are H-equivalent. (The system $b_0, b_1, \dots, b_r$ may be taken fixed.) Thus the number of equivalence classes under this H-equivalence equals the number of non-isomorphic rings over G with $\mathfrak{R}$ as an annihilator, and we conclude:

Theorem 2. *A torsion free group G is a quasi nil group if and only if it is either a nil group or has the form*

$$G = \mathfrak{R} + H$$

where H is a nil group of finite rank r such that the number of classes of H-equivalence in the set of $r \times r$ square matrices[9]) *with rational elements is finite.*

In particular, let us consider the case $r = 1$. Then both Λ and M are rational numbers and we may take $P = 1$ (corresponding to the identity automorphism of H) and then conclude that there are two H-equivalence classes, namely $\Lambda = 0$ alone forms one class and the nonzero rationals form the other class. Thus the group $G = \mathfrak{R} + H$ with a nil group H of rank 1 is a quasi nil group. Over this G the following non-isomorphic rings may be defined:

[9]) Of course, relative to a fixed maximal independent system.

1. the zero ring;

2. over $\mathfrak{R}$ define the rational number field F, and over H a zeroring $\bar{H}$, and take[10]) $F \oplus \bar{H}$ (see Case 1);

3. define $\mathfrak{R}$ to be the annihilator of the ring and the products of the elements of H to lie in $\mathfrak{R}$.

This example disproves a conjecture of SZELE [7] which stated that besides $\mathfrak{R}$ and the nil groups there exist no torsion free quasi nil groups.

§ 4. The case of mixed groups.

Assume G is a mixed quasi nil group. Since Lemma 1 is valid for mixed groups too, by the same argument as in § 2 we may conclude that almost all p-components T_p of the torsion subgroup T of G are divisible groups and those T_p which are not divisible have a finite basic subgroup B_p. Then $T_p = B_p + D_p$ with a divisible group D_p and T is of the type $T = B + D$, B a finite, D a divisible group. By a well-known result, if in a mixed group the (maximal) torsion subgroup is of this type, then it is a direct summand, that is,

(6) $$G = B + D + J$$

where $J \neq 0$ is torsion free. Evidently, J must again be a quasi nil group, hence is of a structure described by Theorem 2.

Next suppose that $D \neq 0$, i. e. in G there exists a direct summand of the type $\mathfrak{C}(p^\infty)$ for some prime p. Then for this prime p necessarily $pJ = J$ holds. In fact, if pJ is a proper subgroup of J, then $p^n J$ is a proper subgroup of $p^{n-1}J$ ($n = 2, 3, \ldots$), and thus there is a homomorphism $J/p^n J \sim \mathfrak{C}(p^n)$ and hence a homomorphism $J \sim \mathfrak{C}(p^n)$. Let $\mathfrak{C}(p^\infty) = \{c_1, c_2, \ldots\}$ with $pc_1 = 0$, $pc_2 = c_1, \ldots$. According to (6), each element g of G has a unique representation $g = b + d + a$ ($b \in B, d \in D, a \in J$). Define a ring R_n over G by the multiplication rule

(7) $$g_1 g_2 = (b_1 + d_1 + a_1)(b_2 + d_2 + a_2) = k_1 k_2 c_n$$

where $k_1 c_n, k_2 c_n$ are the images of a_1, a_2 under $J \sim \mathfrak{C}(p^n)$. Since in R_n any product of three elements vanishes, (7) actually implies a ring R_n over G. Clearly, by $R_n^2 = \{c_n\}$, the R_n are not isomorphic for different n's, thus the hypothesis $pJ \subset J$ contradicts the quasi nil character of G. We have thus proved that the presence of $\mathfrak{C}(p^\infty)$ in G implies $pJ = J$. — Moreover, it follows that the rank of J is 1. In order to verify this assertion, take any two independent elements u, v in J; then each element a of J has the form

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[10]) The sign $\oplus$ will be used to denote direct sum in the ring-theoretic sense.

$a = \varrho u + \sigma v + x$ for some x (which identically vanishes if the rank of J is 2) and rational numbers ϱ, σ. If we agree in putting $\frac{1}{p^k} c_l = c_{k+l}$ in $\mathfrak{C}(p^\infty)$, then $\mathfrak{C}(p^\infty)$ may be regarded as a group with rational operators (for $\varrho c, \varrho$ a rational number, c in $\mathfrak{C}(p^\infty)$, is a well-defined element in $\mathfrak{C}(p^\infty)$). We define, for each p-adic integer π, a ring $R(\pi)$ over G by the rule

$$g_1 g_2 = (b_1 + d_1 + \varrho_1 u + \sigma_1 v + x_1)(b_2 + d_2 + \varrho_2 u + \sigma_2 v + x_2) = (\varrho_1 \varrho_2 + \sigma_1 \sigma_2 \pi) c_1.$$

$R(\pi)$ is plainly a ring. Consider the elements g which are divisible by every power of p. If $\frac{g}{p^\nu}$ denotes any element y with $p^\nu y = g$, then[11])

$$\left(\frac{g}{p^\nu}\right)^2 = \left(\frac{b}{p^\nu} + \frac{d}{p^\nu} + \frac{\varrho}{p^\nu} u + \frac{\sigma}{p^\nu} v + \frac{x}{p^\nu}\right)^2 = \frac{\varrho^2 + \sigma^2 \pi}{p^{2\nu}} c_1 = (\varrho^2 + \sigma^2 \pi) c_{2\nu+1}.$$

Thus every g, divisible by all powers of p, defines an endomorphism of $\mathfrak{C}(p^\infty)$ which may be represented by the p-adic integer $\varrho^2 + \sigma^2 \pi$. The set of these p-adic integers, taken for all g, contains 1 and is countable. If two rings are isomorphic, then the corresponding sets of p-adic integers may differ merely by a p-adic unit factor (inducing an automorphism on $\mathfrak{C}(p^\infty)$). Since 1 was supposed to belong to this set, there is but a countable set of p-adic integers belonging to a class of isomorphic rings. The uncountability of the p-adic integers implies that there is an infinity of non-isomorphic rings $R(\pi)$ over G. Consequently, J must be of rank 1.

Next we show that there is but a finite number of primes p for which $\mathfrak{C}(p^\infty)$ exists in D. For, in the contrary case there would exist a homomorphism η_p of J into each of these $\mathfrak{C}(p^\infty)$, and by the same methods as used in the preceding paragraph we could show that each η_p gives rise to a ring $R(p)$ over G such that all products lie in $\mathfrak{C}(p^\infty)$, but not all of them vanish. Since G is a quasi nil group, this is impossible.

Assume that $G = B + J$ where B is finite and J is a nil group, and let p be a prime dividing the order m of B. Then J/pJ is finite, for in the contrary case there would exist in J an infinite set of independent elements $a_1, a_2, \ldots$ belonging to pairwise different cosets $\bmod pJ$. Let $b \in B$ be of order p and put $a_i^2 = b$ if $i > n$ and $a_i a_j = 0$ in all other cases, furthermore, for the elements independent of the a_i define the multiplication to be identically 0. Then this definition gives rise to a ring R_n over G and for different n's the rings R_n are not isomorphic, for the annihilator of R_n mod $\{B, pJ\}$ is

[11]) For simplicity assume (this can always be done without restricting generality) that in the denominator of ϱ and σ the prime p does not occur.

of rank n (note that B is the torsion subring and pJ is also an invariant for all rings over G). It follows that J/pJ and hence J/mJ is finite.

What we have proved shows that a mixed quasi nil group G has one of the forms

I. $G=B+J$ where B is finite of order m, J a torsion free quasi nil group such that J/mJ is finite whenever J is a nil group.

II. $G=B+D+J$ where B is finite, D a torsion divisible group with a finite number of p-components, J a torsion free quasi nil group of rank 1 such that $pJ=J$ for the primes p occurring in D.

Conversely, assume the group G has the form I. We intend to show that but a finite number of non-isomorphic rings exists over G.

It is evident that mJ annihilates B and among the elements of J only those outside mJ may have a product not belonging to J. In order to know a ring R over G, it suffices to know the following products: 1. the elements of J by the elements of J; 2. the elements of B by the elements of B; 3. the elements of B by some representatives of J mod mJ. The products 2. and 3. lie in B, thus there is but a finite number of possibilities for defining them. The products 1. are of the form $a_1a_2=a_3+b$ $(a_i\in J, b\in B)$; here b does not alter if we replace a_1 and a_2 by other elements of the cosets of a_1 and a_2 mod mJ. Thus to each ring S over $J\cong G/B$ there is but a finite number of rings R over G with $S\cong R/B$. If the rings S_1 and S_2 over J are isomorphic, and R_1 is a ring over G which corresponds to S_1, then we may extend S_2 such that the B-components of the products in 2. and 3. be the same in R_2 as those of the corresponding elements in R_1 (we let B fixed). To be more explicit, if e. g. $a_1a_2=a_3+b$ $(a_i\in J, b\in B)$ in R_1, then $a_1a_2=a_3$ holds in S_1, and if φ is an isomorphism of S_1 onto S_2, then we set $a_1^\varphi a_2^\varphi=a_3^\varphi+b$. It is easily seen that, since mJ is carried onto itself by every automorphism, the rings R_1 and R_2 will be isomorphic, and this establishes what we intended to verify in this paragraph.

Let now G have the form II and consider those rings R over G in which all the products lie in the torsion subgroup $B+D$ of G. First of all observe that D is an annihilator of G, for besides it annihilates $B+D$, it so does J, considering that $pJ=J$ holds for all p with $\mathfrak{C}(p^\infty)\subseteq D$.

For a fixed $u\in J$, the mapping $v\rightarrow uv$ is a homomorphism of J onto a subgroup T_u of $B+D$, and from $r(J)=1$ we conclude that T_u has the form[12]

$$T_u=\mathfrak{C}(p_1^\infty)+\cdots+\mathfrak{C}(p_s^\infty)+\mathfrak{C}(q_1^{k_1})+\cdots+\mathfrak{C}(q_t^{k_t}) \tag{7}$$

with different primes $p_1,\ldots,p_s,q_1,\ldots,q_t$. If $pJ=J$, then also $pT_u=T_u$

12) See e. g. Beaumont and Zuckerman [1].

so that $q_j J \neq J$ and therefore $\mathfrak{C}(q_j^\infty)$ does not exist in D, i. e. the "finite part" $\mathfrak{C}(q_1^{k_1}) + \cdots + \mathfrak{C}(q_t^{k_t})$ of T_u belongs to B. Choose a $u \in J$ such that u is not divisible by those primes q of the order of B for which $qJ \neq J$, and no p_i-component of u^2 in T_u is zero $(i = 1, \ldots, s)$. Then the squares u^2/p_i^{2n} already determine all products vw $(v, w \in J)$, since $v = \varrho u$, $w = \sigma u$ with rational ϱ, σ and thus $vw = \varrho\sigma u^2$ is a well-defined element of T_u whenever in T_u the multiplication by rationals is appropriately defined. Next take into account that the multiplications of J by $p_1, \ldots, p_r$, respectively define automorphisms of J, so that only the fact is essential that the components of the squares $v^2 (v \in J)$ in $\mathfrak{C}(p_1^\infty), \ldots, \mathfrak{C}(p_s^\infty)$, respectively, are of odd or even exponents. Consequently, there is but a finite number of possibilities for defining the multiplication of the elements of J in order to obtain non-isomorphic rings. The same holds for the products $b_1 b_2 (b_i \in B)$ and the products of the elements of B by representatives of $J \bmod mJ$, since it is irrelevant which subgroup of type $\mathfrak{C}(p^\infty)$ in D will contain components of products. It results that over a group of type II there exists but a finite number of non-isomorphic rings with products in the torsion subgroup.

Let G be again of type II and consider the case when not all the products lie in the torsion subgroup $B + D$. Then the factor ring with respect to the ideal $B + D$ is not a zero ring, consequently, J must be isomorphic to $\mathfrak{R}$. Now in any ring R over G the products $a_1 a_2 (a_i \in J)$ are divisible by every integer, thus they belong to $D + J$ (the maximal divisible subgroup of G). It is not hard to verify that $(\varrho a) a = \varrho a^2$ varies over a subgroup K of G, $K \cong \mathfrak{R}$, when a is fixed in J and ϱ runs over all rationals. Then any product $g_1 g_2$ with $g_i \in K$ lies in K and $B + D$ must belong to the annihilator of K, consequently, K is a direct summand of R in the ring-theoretic sense: $R = (B + D) \oplus K$. Since $B + D$ is a quasi nil torsion group and the ring over K is isomorphic to the rational number field, we arrive at the following result.

Theorem 3. *A mixed group G is a quasi nil group if and only if it is either of the form I or of the form II.*

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