On relatively complemented lattices.

By G. SZÁSZ in Szeged.

1. Throughout this paper let L denote a relatively complemented lattice with greatest and least elements i, o, respectively[1]). Let further a, b, r be any elements of L such that

$$a \leqq r \leqq b. \tag{1}$$

As usual, by a *relative complement of r in* $[a, b]$ we mean an element s which satisfies the equations

$$r \cap s = a, \quad r \cup s = b. \tag{2}$$

Clearly, s then also belongs to the interval $[a, b]$.

J. v. NEUMANN has proved[2]) that if L is modular, then, for any complement t of r, the element

$$s = (a \cup t) \cap b = a \cup (t \cap b) \tag{3}$$

is a relative complement of r in $[a, b]$. It is known that this theorem plays a very important role in the theory of modular lattices.

In this paper we shall establish further connections between the complements and relative complements of an element r of L.

2. First we state, without assuming the modularity, the following converse of NEUMANN's Theorem:

Theorem 1. *Let L be any relatively complemented lattice with greatest and least elements, and let a, b, r be any elements of L such that* (1) *holds. Let further s be any relative complement of r in* $[a, b]$. *Then there exists at least one complement t of r which satisfies* (3).

---

[1]) For the concepts of lattice theory which will not be defined and for the results which will be used without proof in this paper, see G. BIRKHOFF, *Lattice theory* (Amer. Math. Soc. Coll. Publ., vol. 25), revised edition, New York, 1948.

[2]) See, for example, G. BIRKHOFF, op. cit., p. 114. References to this theorem will be made below briefly by the term "NEUMANN's Theorem".

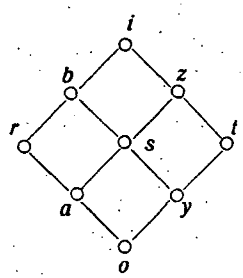
This theorem is an immediate corollary of the second part of the more general

Theorem 2. *Let L, a, b, r, s be as in Theorem 1 and let t $(\in L)$ be any solution of the equation system*

$$(4)\qquad \begin{cases} r\cap t=o, \\ r\cup t=i, \\ (a\cup t)\cap b=s, \\ a\cup(t\cap b)=s. \end{cases}$$

Then there exists a relative complement y of a in $[o,s]$ and a relative complement z of b in $[s,i]$ such that t is a relative complement of s in $[y,z]$.

Conversely, if y is any relative complement of a in $[o,s]$ and z is any relative complement of b in $[s,i]$, then any relative complement t of s in $[y,z]$ satisfies the equation system (4). (See the figure.)



Proof. In order to prove the first part of Theorem 2, let us consider any solution t of (4) and let us define two elements y,z by

$$(5)\qquad y=s\cap t,\quad z=s\cup t.$$

Then, by the choice of these elements, t is a relative complement of s in $[y,z]$. Furthermore, by the last two equations of (4), we have

$$(6)\qquad y=s\cap t=(a\cup t)\cap b\cap t=b\cap t,$$

$$(7)\qquad z=s\cup t=a\cup(t\cap b)\cup t=a\cup t.$$

We show that

$$(8)\qquad a\cap y=o,\quad a\cup y=s$$

and

$$(9)\qquad b\cap z=s,\quad b\cup z=i.$$

Indeed, (6), (1) and the first equation of (4) imply

$$a\cap y=a\cap(b\cap t)=(a\cap b)\cap t=a\cap t\leqq r\cap t=o,$$

and (6) and the last equation of (4) imply

$$a \cup y = a \cup (b \cap t) = s.$$

Similarly, by (7), (1) and (4), we obtain (9). Clearly, by (5), (8) and (9), the first statement of our theorem is proved.

Conversely, let y, z, t be any elements satisfying the equations (5), (8) and (9). Then, firstly, t is a complement of r. Indeed, by (1), (5), (9), (2), (5) and (8),

$$r \cap t = (r \cap b) \cap (z \cap t) = r \cap (b \cap z) \cap t =$$
$$= r \cap s \cap t = (r \cap s) \cap (s \cap t) = a \cap y = o,$$

and dually,

$$r \cup t = i.$$

Moreover, t satisfies the last two equations of (4). For by (5), (8), (5) and (9)

$$(a \cup t) \cap b = (a \cup (y \cup t)) \cap b = ((a \cup y) \cup t) \cup b = (s \cup t) \cap b = z \cap b = s,$$

and by (5), (9), (5) and (8)

$$a \cup (t \cap b) = a \cup ((t \cap z) \cap b) = a \cup (t \cap (z \cap b)) = a \cup (t \cap s) = a \cup y = s,$$

thus completing the proof.

By Theorems 1 and 2 we have the following

Corollary. *Let L, a, b, r, s be as in Theorem* 1. *Then, by suitable choice of the complements a', b', s' of a, b, s, respectively, each solution t of* (4) *may be represented in the form*

$$t = ((a' \cap s) \cup s') \cap (s \cup b') = (a' \cap s) \cup (s' \cap (s \cup b')). \tag{10}$$

Proof. Let t be any solution of (4) and let y, z be defined as in the proof of the first part of Theorem 2. Then, with regard to the equations (5), (8) and (9), Theorem 1 implies that for some complements a', b', s' of a, b, s, respectively,

$$y = o \cup (a' \cap s) = a' \cap s,$$
$$z = (s \cup b') \cap i = s \cup b',$$
$$t = (y \cup s') \cap z = y \cup (s' \cap z).$$

These representations obviously yield the corollary.

3. This section will be concerned with the special case when L is modular. We recall the reader that, by NEUMANN's Theorem, complemented modular lattices are also relatively complemented; consequently, Theorem 1 and 2 may be applied for them.

Using the results of the preceding section, we prove

Theorem 3. *Let L be any complemented modular lattice and let a, b, r be any elements of L satisfying* (1). *Then, s being any relative complement* *of r in $[a, b]$ and a', b', s' being arbitrary complements of a, b, s, respectively, the element t of the form* (10) *is a complement of r.*

Conversely, to each complement t of r there exists at least one relative complement s of r in $[a, b]$ such that, by suitable choice of the complements a', b', s' of a, b, s, respectively, the equation (10) *is satisfied*[3]).

Proof. Let s denote any relative complement of r in $[a, b]$. Consider the elements

$$\begin{aligned} y &= o \cup (a' \cap s) = a' \cap s, \\ z &= (s \cup b') \cap i = s \cup b', \\ t &= (y \cup s') \cap z = ((a' \cap s) \cup s') \cap (s \cup b') = \\ &= y \cup (s' \cap z) = (a' \cap s) \cup (s' \cap (s \cup b')), \end{aligned}$$

where a', b', s' denote arbitrary complements of a, b, s, respectively. Then, by NEUMANN's Theorem,

1. y is a relative complement of a in $[o, s]$;
2. z is a relative complement of b in $[s, i]$;
3. t is a relative complement of s in $[y, z] (= [a' \cap s, s \cup b'])$.

Hence, by the second part of Theorem 2, t is a complement of r, as asserted.

Conversely, if t is a complement of r, then, again by NEUMANN's Theorem, the element s of the form (3) is a relative complement of r in $[a, b]$. It follows that, for this s, the element t is a solution of (4). Hence, by the Corollary obtained in the preceding section, we conclude that, with some complements a', b', s' of a, b, s, respectively, the element t may be represented in the form (10). This completes the proof of Theorem 3.

(*Received April 8, 1957.*)

[3]) The first part of this theorem may be proved also by a direct, but very tedious calculation.