A theorem on algebraic operators in the most general sense.

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Let be given the sets H and $\mathfrak{H}$. Let be defined an operation $a\alpha$ ($a \in H$, $\alpha \in \mathfrak{H}$) with values in H (i. e. a function which assignes to each pair of elements $a \in H$, $\alpha \in \mathfrak{H}$ an element of H).

For any subset $\mathfrak{G}$ of $\mathfrak{H}$ we denote by $S(\mathfrak{G})$ the set of elements of H which remain fixed by all elements of $\mathfrak{G}$.

Let be chosen a subset $\mathfrak{A}$ of $\mathfrak{H}$. For any subset G of H we denote by $\mathfrak{S}_{\mathfrak{A}}(G)$ the set of elements of $\mathfrak{A}$ which leave fixed all the elements of G.[1])

The proof of our theorem will be simplified if we anticipate some statements about the concepts already exposed. (Some of these statements are well known in Galois theory in particular cases.)

α) For each $G\,(\subseteqq H)$ and $\mathfrak{A}\,(\subseteqq \mathfrak{H})$ we have $G \subseteqq S(\mathfrak{S}_{\mathfrak{A}}(G))$.

β) $\mathfrak{A} \supseteqq \mathfrak{G}$ implies[2]) $\mathfrak{S}_{\mathfrak{A}}(S(\mathfrak{G})) \supseteqq \mathfrak{G}$.

γ) $\mathfrak{G}_1 \supseteqq \mathfrak{G}_2$ implies $S(\mathfrak{G}_1) \subseteqq S(\mathfrak{G}_2)$.

δ) $G_1 \supseteqq G_2$ implies $\mathfrak{S}_{\mathfrak{A}}(G_1) \subseteqq \mathfrak{S}_{\mathfrak{A}}(G_2)$.

ε) $\mathfrak{A} \supseteqq \mathfrak{B}$ implies $\mathfrak{S}_{\mathfrak{A}}(G) \supseteqq \mathfrak{S}_{\mathfrak{B}}(G)$.

η) $\mathfrak{B} \supseteqq \mathfrak{S}_{\mathfrak{A}}(G)$ implies $\mathfrak{S}_{\mathfrak{B}}(G) \supseteqq \mathfrak{S}_{\mathfrak{A}}(G)$.

These statements require no proofs, exceptly perhaps η) which can be verified by intersecting both sides of the supposition by $\mathfrak{S}_{\mathfrak{H}}(G)$.

Definition 1. The subset G of H is $\mathfrak{A}$-replete if $G = S(\mathfrak{S}_{\mathfrak{A}}(G))$ holds.

Definition 2. The subset $\mathfrak{G}$ of $\mathfrak{H}$ is $\mathfrak{A}$-replete if $\mathfrak{G} = \mathfrak{S}_{\mathfrak{A}}(S(\mathfrak{G}))$ holds.[3])

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[1]) Hence we have $\mathfrak{S}_{\mathfrak{A}}(G) = \mathfrak{A} \cap \mathfrak{S}_{\mathfrak{H}}(G) \subseteqq \mathfrak{A}$ for any $\mathfrak{A} \subseteqq \mathfrak{H}$ and $G \subseteqq H$.

[2]) We denote always the subsets of H by Roman capital letters and the subsets of $\mathfrak{H}$ by German capital ones. (The letters S and $\mathfrak{S}$ serve other purpose.)

[3]) Compare these definitions to the similar ones in the paper G. BIRKHOFF, On the structure of abstract algebras, *Proc. Cambridge Phil. Soc.*, 31 (1935), 433–454, especially p. 435.

Theorem. *Let be given the subsets* $\mathfrak{A}$ *and* $\mathfrak{B}$ *of* $\mathfrak{H}$ *for which we have* $\mathfrak{A} \supseteqq \mathfrak{B}$. *In this case the following four statements are true:*

1. *If* G *is* $\mathfrak{B}$*-replete, then* G *is* $\mathfrak{A}$*-replete.*

2. *If* G *is* $\mathfrak{A}$*-replete and* $\mathfrak{B}$ *is* $\mathfrak{A}$*-replete and* $G \supseteqq S(\mathfrak{B})$, *then* G *is* $\mathfrak{B}$*-replete.*

3. *If* $\mathfrak{G}$ *is* $\mathfrak{A}$*-replete and* $\mathfrak{G} \subseteqq \mathfrak{B}$, *then* $\mathfrak{G}$ *is* $\mathfrak{B}$*-replete.*

4. *If* $\mathfrak{G}$ *is* $\mathfrak{B}$*-replete and* $\mathfrak{B}$ *is* $\mathfrak{A}$*-replete, then* $\mathfrak{G}$ *is* $\mathfrak{A}$*-replete.*

Proof. 1. We have

$$G \subseteqq S(\mathfrak{S}_{\mathfrak{a}}(G)) \subseteqq S(\mathfrak{S}_{\mathfrak{B}}(G)) = G.$$

The first inclusion is the statement α), the second one is implied by ε) and γ), and the equality by the supposition.

2. The supposed inclusion and δ) imply $\mathfrak{S}_{\mathfrak{a}}(G) \subseteqq \mathfrak{S}_{\mathfrak{a}}(S(\mathfrak{B}))$. Hence, $\mathfrak{B}$ being $\mathfrak{A}$-replete, we have $\mathfrak{S}_{\mathfrak{a}}(G) \subseteqq \mathfrak{B}$. Therefore, by ε) and η) we have $\mathfrak{S}_{\mathfrak{a}}(G) = \mathfrak{S}_{\mathfrak{B}}(G)$, thus $S(\mathfrak{S}_{\mathfrak{B}}(G)) = S(\mathfrak{S}_{\mathfrak{a}}(G)) = G$.

3. We have

$$\mathfrak{G} \subseteqq \mathfrak{S}_{\mathfrak{B}}(S(\mathfrak{G})) \subseteqq \mathfrak{S}_{\mathfrak{a}}(S(\mathfrak{G})) = \mathfrak{G}.$$

The first inclusion is implied by the statement β), the second one by ε), and the equality by the supposition.

4. We get

$$\mathfrak{S}_{\mathfrak{a}}(S(\mathfrak{G})) \subseteqq \mathfrak{S}_{\mathfrak{a}}(S(\mathfrak{B}))$$

applying γ) and δ) to the inclusion $\mathfrak{G} = \mathfrak{S}_{\mathfrak{B}}(S(\mathfrak{G})) \subseteqq \mathfrak{B}$. Owing to $\mathfrak{S}_{\mathfrak{a}}(S(\mathfrak{B})) = \mathfrak{B}$ we can apply η) for $G = S(\mathfrak{G})$, hence and by ε) we have the equality

$$\mathfrak{S}_{\mathfrak{a}}(S(\mathfrak{G})) = \mathfrak{S}_{\mathfrak{B}}(S(\mathfrak{G})) = \mathfrak{G}.$$

Remark. (On November 28, 1957.) Let be defined the mappings $\mathfrak{G} \to S_A(\mathfrak{G})$ $(= S(\mathfrak{G}) \cap A)$ for any subset A of H, and the mapping $G \to \mathfrak{S}(G)$ $(= \mathfrak{S}_{\mathfrak{H}}(G))$. In this case we can deduce results which are in duality relation to the above ones.

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