A simple proof of the functional equation for the Riemann zeta-function and a formula of Hurwitz.

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1. The functional equation of $\zeta(s)$ as a fundamental fact of the analytic theory of numbers was discussed since RIEMANN many times and in various ways. Besides the two original proofs of RIEMANN which are based on the contour integral representation of the zeta-function and on certain properties of the theta-function $\Sigma e^{-n^2\pi x}$, respectively, new proofs were published in the last decades e. g. by HARDY, INGHAM, KLOOSTERMAN, MORDELL, RADEMACHER, SIEGEL, TITCHMARSH.[1])

In the present paper we shall give a remarkably simple proof consisting substantially of the calculation of certain Fourier coefficients by the formula

$$\int_0^\infty t^{z-1} e^{-i\lambda t}\, dt = \Gamma(z)\,(i\lambda)^{-z} \qquad (\lambda \neq 0 \text{ real},\ 0 < \Re(z) < 1), \tag{1}$$

which is an immediate consequence of EULER's integral definition for $\Gamma(z)$[2]). It is easy and appropriate in addition to verify more: the possibility of analytic continuation over the whole s-plane of the generalized zeta-function $\zeta(s, u) = \sum_{n=0}^{\infty} (u+n)^{-s}$ $(\Re(s) > 1,\ 0 < u \leqq 1)$ and the formula of HURWITZ:[3])

$$\zeta(s, u) = 2\Gamma(1-s) \sum_{\nu=1}^{\infty} (2\nu\pi)^{s-1} \sin\left(2\nu\pi u + \frac{\pi s}{2}\right), \tag{2}$$

[1]) Cf. e. g. E. LANDAU, *Vorlesungen über Zahlentheorie* (Leipzig, 1927), and E. C. TITCHMARSH, *The theory of the Riemann zeta-function* (Oxford, 1951). The latter contains also an extended list of original papers.

[2]) The integral in (1) extends over the positive real axis and $(i\lambda)^{-z}$ is to be taken with its principal value.

[3]) See E. T. WHITTAKER and G. N. WATSON, *Modern analysis*, 4. edition (Cambridge, 1952), p. 266—267. — We shall show (without using any integral representation of $\zeta(s, u)$) the validity of (2) for $\Re(s) < 1$, while one finds it only with the condition $\Re(s) < 0$ in the literature. (Cf. to this also certain remarks of my two recent papers in *these Acta*, **17** (1956), 143—164, and in *Publ. Math. Debrecen*, **5** (1957), 44—53, where (2) occurs as an auxiliary tool.)

which furnishes for $u=\frac{1}{2}$ or 1 by $\zeta\left(s,\frac{1}{2}\right)=(2^s-1)\zeta(s)$, $\sum_{\nu=1}^{\infty}(-1)^\nu\nu^{-s}=$ $=(2^{1-s}-1)\zeta(s)\,(\Re(s)>0)$ and $\zeta(s,1)=\zeta(s)$, respectively, the relation in question:

$$(2^*)\qquad \zeta(s)=2\,(2\pi)^{s-1}\,\Gamma(1-s)\sin\frac{\pi s}{2}.\,\zeta(1-s).$$

By (2) one can get at once the corresponding facts in the theory of DIRICHLET's L-functions, first of all their functional equation[4]).

2. Theorem. $\zeta(s,u)$ *with any fixed* u *in* (0,1] *is regular for all* s *except for* $s=1$; *at this point there is a simple pole with residue* 1. *Formula* (2) *holds for* $0<u<1$, $\Re(s)<1$, *in particular formula* (2*) *holds for all* $s\neq 1,2,\dots$.

Proof. 1° Let $\Re(s)>0, s\neq 1$. Then the difference $\sum_{n=1}^{N}(u+n)^{-s}-$ $-\frac{1}{1-s}N^{1-s}$ $(0\leqq u\leqq 1)$ converges as $N\to\infty$ and its limit is an analytic function of s, since we have the estimation

$$(^*)\qquad \left|(u+n)^{-s}-\frac{1}{1-s}\left[n^{1-s}-(n-1)^{1-s}\right]\right|\leqq\frac{3}{2}Rn^{-1-\varepsilon}\left[1+\frac{R+1}{2}\frac{1}{n}+\right.$$
$$\left.+\frac{(R+1)(R+2)}{2\cdot 3}\frac{1}{n^2}+\cdots\right]\leqq 3Rn^{-1-\varepsilon},$$

provided that $\Re(s)\geqq\varepsilon>0$, $|s|\leqq R$, $n\geqq 2R>2$. Therefore for $0<u\leqq 1$

$$(3)\qquad \zeta(s,u)=\lim_{N\to\infty}\left[\sum_{n=0}^{N}(u+n)^{-s}-\frac{N^{1-s}}{1-s}\right],$$

$$(4)\qquad \zeta(s,u)-\frac{1}{s-1}=u^{-s}+(u+1)^{-s}+\sum_{n=2}^{\infty}\left[(u+n)^{-s}-\frac{n^{1-s}-(n-1)^{1-s}}{1-s}\right]$$

The sum of the last series being regular for $\Re(s)>0$ by (*), (4) implies at once the assertion concerning $s=1$.

2° Next consider $\sum_{\nu=1}^{\infty}\nu^{-s}e^{2\nu\pi iu}$ $(0<u<1)$. By partial summation and since $|\nu^{-s}-(\nu+1)^{-s}|\leqq 6R\nu^{-1-\varepsilon}$ $(\Re(s)\geqq\varepsilon>0,\ |s|\leqq R,\ \nu\geqq 2R>2)$ (cf. (*)), this series converges in the half-plane $\Re(s)>0$ and represents there a regular function of s. It results the same behaviour for $\Re(s)<1$ of the Hurwitz series in (2).

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[4]) Cf. Н. Чудаков, Введение в теорию L-функции Дирихлет (Moscow—Leningrad, 1947), p. 89 and 92.

3° Now it remains only to show that (2) is valid e. g. for $0 < s < 1$, $0 < u < 1$. In fact, the complex Fourier coefficients of $\zeta(s, u)$ with such s, u are by (3) and (1):

$$c_0 = \int_0^1 \zeta(s, u)\, du = 0,$$

$$c_\nu = \int_0^1 \zeta(s, u)\, e^{-2\nu\pi i u}\, du = \int_0^\infty v^{-s} e^{-2\nu\pi i v} dv = \Gamma(1-s)(2\nu\pi i)^{s-1} \quad (\nu = \pm 1, \pm 2, \ldots),$$

the term-by-term integration being justified by (*); hence the right-hand series in (2) is the (real) Fourier development of $\zeta(s, u)$ over $0 < u < 1$.

Since, by the uniform convergence for $0 \leqq u \leqq 1$ of $\sum_{n=1}^{\infty} (u+n)^{-1-s}$, we have $\frac{d}{du}\zeta(s, u) = -s\,\zeta(s+1, u)$ in this interval, we conclude at the same time to its convergence to the function in question, q. e. d.

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