On ideal theory for lattices.

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1. Introduction.

The notion of lattice ideals plays an important role in lattice-theoretical researches. Recently J. HASHIMOTO [5] developed a theory of lattice ideals making effort to evolve this algebraic theory like that of rings. The fundamental tools of HASHIMOTO's paper are various topologies of lattices. Consequently, his purely lattice-theoretical assertions too are mostly proved with the apparatus of topology, so these proofs are not quite short and it is not easy to follow them.

The aim of the present paper is to prove in purely lattice-theoretical ways all purely lattice-theoretical theorems of [5]. These proofs are generally more concise than the original ones, further they offer more generalizations. In this paper we shall not deal with these generalizations. Related to these questions we refer to the papers [2], [3], [4].

In building up our paper we adhered strictly to the structure of HASHIMOTO's paper; the titles of the parts beginning from 3 — as well as the greatest part of the terminology — are identical with that of [5].

We should mention at last that all of the theorems, not containing explicitly the existence of prime ideals may be proved also without the Axiom of Choice (we hint here in the first line to the results in connexion with problems 72 and 73 of G. BIRKHOFF [1]). As for these proofs we refer to our above cited papers [2] and [3].

2. Some lemmas.

Before turning to HASHIMOTO's theorems we mention some lemmas in advance.

Lemma I (STONE's Theorem). *Let L be a distributive lattice and let I and D be any ideal resp. dual ideal of L such that I and D are disjoint.*

Among the ideals which are disjoint to D and contain I every maximal one is prime.[1])

The proof of this lemma runs along the same lines as the proof of the original assertion of STONE (see e. g. [1], p. 160).

Lemma II. *Let L be a distributive lattice. If the meet and the join of the ideals I and J are principal ideals, then I and J are principal ideals too.*

Proof. Let $I \cup J = (a]$ and $I \cap J = (b]$. It follows from the distributivity of L ([1], pp. 140—141) that for some $x, y \in I$ and $u, z \in J$, $x \cup z = a$ and $y \cap u = b$. It is easy to check that $(x \cup y) \cup (z \cup u) = a$ and $(x \cup y) \cap (z \cup u) = b$. We assert $I = (x \cup y]$ and $J = (z \cup u]$. If we had e. g. $I \neq (x \cup y]$, then there would exist in I an element w such that $x \cup y < w$. But then $w \cup (z \cup u) = a$ and $w \cap (z \cup u) = b$, that is, $z \cup u$ has two relative complements in the interval $[b, a]$, namely $x \cup y$ and w. It is well known that in a distributive lattice any element cannot have in every interval more than one relative complement, which contradicts the fact proved above and completes the proof of this lemma.

Lemma III. *Let L' be any homomorphic image of the lattice L and P' a prime ideal of L'. The complete inverse image of P' in L is a prime ideal.*

Proof. Let P be the complete inverse image of P'; evidently, P is an ideal. We prove that P is prime. Let us suppose that $x, y \notin P$, yet $x \cap y \in P$. Then, if we denote by x' and y' the homomorphic image of x and y, respectively, we get that x' and y' are not in P', for P is the complete inverse image of P'. Furthermore, $x' \cap y' = (x \cap y)' \in P'$, contradicting the assumption that P' is a prime ideal. Thus the proof is completed.

3. Possibility of factorization.

By a *representation* of a lattice L we mean here a homomorphism of L onto a distributive lattice. (This definition is not the same as, but is equivalent to, that of [5].)

Theorem I (Theorem 2.1 of [5]). *The following assertions concerning an ideal I of a lattice L are equivalent:*

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1) This is a somewhat generalized form of STONE's Theorem, namely STONE restricts himself to the case when the dual ideal D is principal. The above form of the theorem has the advantage that every prime ideal P may be constructed in such a way (with $I = P$ and $D = L - P$), while originally only the completely meet-irreducible prime ideals could be constructed (as it is an easy consequence of a result of G. BIRKHOFF and O. FRINK).

(1) *I is the intersection of the prime ideals which contain it; in other words, I is the product of all its prime ideal divisors;*

(2) *I is the kernel of some representation.*

Proof. Firstly we verify that (1) implies (2). Let I be the intersection of all prime ideals which contain it, $I = \wedge P_\alpha$. There exists to every prime ideal P_α a congruence relation Θ_α with the property that[2]) $L(\Theta_\alpha) \cong \mathbf{2}$ and P_α is a congruence class under Θ_α. Obviously, the kernel of the congruence relation $\Theta = \wedge \Theta_\alpha$ is I. Hence it remains to prove that Θ is a representation, i. e. that $x \cup (y \cap z) \equiv (x \cup y) \cap (x \cup z)\, (\Theta)$. But this is evident, for $a \equiv b(\Theta)$ if and only if $a \equiv b\,(\Theta_\alpha)$ for all α.[3])

On the other hand, let I be the kernel of a representation Θ. The lattice $L(\Theta)$ is distributive and has zero element. Using Lemma I, for all $0 \neq a \in L(\Theta)$ we may construct in $L(\Theta)$ a prime ideal which is disjoint to the dual ideal $[a)$. The meet of these prime ideals is the zero of $L(\Theta)$. I being the complete inverse image of the zero of $L(\Theta)$, the intersection of the complete inverse images of all above constructed prime ideals is I. By Lemma III, the complete inverse image of a prime ideal is again a prime ideal, so we get that I is the intersection of prime ideals. Qu. e. d.

Corollary 1. *Every ideal of a distributive lattice is the product of all its prime ideal divisors.*

Corollary 2. *Every maximal ideal of a distributive lattice is prime.*

(As a matter of fact these Corollaries are immediate consequences already of Lemma I, for any ideal I of the distributive lattice L is the meet of all P_α, $a \notin I$, if P_a is defined as a maximal ideal with $P_a \supseteq I$, $a \notin P_a$; by Lemma I any P_a is prime, hence the assertion follows.) Now we prove the converse of Corollary 1 of Theorem I.

Theorem II (Theorem 2.2 of [5]). *Each of the following conditions is necessary and sufficient in order that a lattice L be distributive:*

(1) *every ideal of L is the intersection of the prime ideals which contain it;*

(2) *every principal ideal of L is the intersection of the prime ideals which contain it;*

(3) *every ideal of L is the kernel of some homomorphism;*

(4) *every principal ideal of L is the kernel of some homomorphism.*

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[2]) $L(\Theta)$ denotes the homomorphic image of L induced by Θ; $\mathbf{2}$ is the lattice of two elements. It is evident that in §§ 2—4 the whole lattice is considered as a prime ideal, but in §§ 5—9 it is not.

[3]) See [1], pp. 23—24.

Proof. With respect to Theorem I and to its Corollary 1, we need only prove that (4) implies the distrubitivity of L. If (4) is valid in L, but L is not distributive, then the latter fact implies that L has a sublattice isomorphic to the lattice of Fig. 1 or of Fig. 2. But in both cases the principal ideal $(a]$ is the kernel of no homomorphism. For, if we suppose that $(a]$ is a congruence class under the congruence relation Θ, then it follows $b = e \cap b = b \cap (a \cup c) \equiv b \cap (d \cup c) = b \cap c = d \ (\Theta)$, but $b \notin (a]$, which is a contradiction.

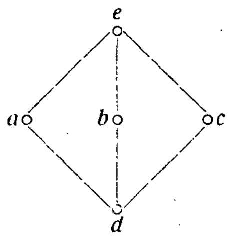

Fig. 1.

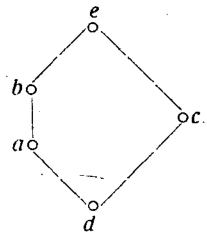

Fig. 2.

Corollary. *Let L be a lattice satisfying the ascending chain condition. L is distributive if and only if $(a]$ is prime for any meet-irreducible element a of L.*

Proof. In consequence of the ascending chain condition, every principal ideal is the meet of a finite number of principal ideals, generated by meet-irreducible elements. If we assume that every principal ideal with meet-irreducible generating element is prime, then we conclude that in L every principal ideal is the intersection of the prime ideals which contain it, i. e. by Theorem II, L is distributive. On the other hand if L is distributive and a is meet-irreducible, then as it is known, $(a]$ is prime (for if $x \cap y \in (a]$, then $a = (x \cap y) \cup a = (x \cup a) \cap (y \cup a)$, that is, $x \cup a = a$ or $y \cup a = a$, i. e. x or $y \in (a]$).

4. Characterization of the lattice of factorizable ideals.

In what follows $\mathfrak{L}$ denotes the lattice of all ideals of the lattice L.

Theorem III (Theorem 3.6 of [5]). *In case of a relatively complemented lattice L the ideals which are the intersections of prime ideals form a dual ideal $\mathfrak{A}$ of $\mathfrak{L}$.*

Proof. Let I be an ideal of the relatively complemented lattice L which is prime factorizable, that is, the intersection of the prime ideals which contain it and K, J two ideals of L with $K \supset J \supseteq I$. By Theorem I, there exists a homomorphic image L' of L with kernel I, such that L' is distributive. We show that if K' and J' denote the homomorphic images of K and J, respectively, then under this homomorphism $K' \neq J'$. The case $J = I$ is trivial. If $J \supset I$, then let us choose three elements a, b, c such that $a \in K - J$ and $c \in J - I$,[4]) $b \in I$. Without loss of generality we may assume that $b < c < a$ (for $b < b \cup c < a \cup b \cup c$ and $b \cup c \in J - I$, $a \cup b \cup c \in K - J$). We denote by $\bar{c}$ any relative complement of c in the interval $[b, a]$. Then $\bar{c} \in K - J$, for in case $\bar{c} \in J$, $a = c \cup \bar{c} \in J$ would be valid too. Now if $K' = J'$, then with suitably chosen a and c, $a \equiv c$ would be valid. It follows $\bar{c} \equiv b$, that is, I would not be the kernel of this homomorphism.

Thus we have proved that every ideal which contains I is the complete inverse image of its homomorphic image in L'. As every ideal of L' is prime factorizable, the same is valid for all complete inverse images of them (see the proof of Theorem I).

On the other hand, if the ideals I and J are prime factorizable, then $I \cap J$ is obviously prime factorizable, completing the proof.

Corollary. *If a relatively complemented lattice has an element a such that both* $(a]$ *and* $[a)$ *are prime factorizable, then L is distributive.*

Proof. It is known that every convex sublattice of L is the set-theoretical intersection of an ideal I und a dual ideal J. If a convex sublattice contains a, then I contains $(a]$ and J contains $[a)$. It follows, by Theorem III, that I is prime factorizable. Consequently, by Theorem I, I is a congruence classe under some congruence relations. Dually we get that J has the same property. Hence, the set-theoretical intersection of I and J is also a congruence classe under a suitable congruence relation.

We obtain that every convex sublattice containing a is a congruence class under one and only one homomorphism. Thus this Corollary is a part of the following Theorem of [2] and [3]:

Le L be a lattice and x a fixed element in L. In order that every convex sublattice of L which contains x be a congruence class under one and only one congruence relation it is necessary and sufficient that L be distributive and every interval $[x, y]$ *or* $[y, x]$ *as a sublattice be complemented.*

[4]) $A - B$ denotes the set-theoretical difference of the sets A and B.

5. Uniqueness of factorization.

If $I=\wedge P_\alpha$, where every P_α is a prime ideal, then it is called a factorization of I (naturally we suppose that if $\alpha\neq\beta$ then $P_\alpha\neq P_\beta$).

Theorem IV (Theorem 4.1 of [5]). *Let L be any lattice and I an ideal of L. If I is represented as an intersection of a finite number of prime ideals, then its irredundant*[5]) *factorization is unique.*

Proof. Let

$$(*)\qquad I=P_1\cap P_2\cap\cdots\cap P_n=Q_1\cap Q_2\cap\cdots\cap Q_k$$

be two irredundant factorizations of I. Let us consider the ideal $J=Q_1\cap\cdots\cdots\cap Q_{i-1}\cap Q_{i+1}\cap\cdots\cap Q_k$. (We have supposed $k>1$. The case $n=k=1$ is obvious.) The factorizations (*) being irredundant, $J\supset I$, that is, there exists an $a\in J$ such that $a\notin I$. Consequently, for at least one index j, $a\notin P_j$. For any $q\in Q_i$ we have $q\cap a\in Q_i\cap J=I\subseteq P_j$, but $a\notin P_j$, therefore $q\in P_j$ (for P_j is prime), that is, $Q_i\subseteq P_j$. In a similar way may be proved the existence of some Q_m such that $P_j\subseteq Q_m$, that is, $Q_i\subseteq P_j\subseteq Q_m$. Since the factorizations (*) are irredundant, this is possible only in case $Q_i=P_j=Q_m$, that is if $i=m$. This at once implies that the two factorizations in (*) are the same.

In proving the Corollary of Theorem II we have seen that in a distributive lattice L a principal ideal $(a]$ is prime if and only if a is meet-irreducible (see [1], p. 142, too). Thus we have the following

Corollary. *In a distributive lattice L, the representation of an element as an irredundant meet of meet-irreducible elements is unique.*

Theorem V (Theorem 4.2 of [5]). *The following statements concerning a distributive lattice L are equivalent:*

(1) *L is relatively complemented;*

(2) *if an ideal of L is decomposed into the product of a finite number of its prime ideal divisors, then this factorization is unique;*

(3) *every prime ideal is maximal;*

(3′) *every dual prime ideal is maximal.*

Proof. (2) *implies* (3). If the prime ideal P is not maximal, then there exists an ideal I with $P\subset I\neq L$. By Lemma 1 there exists a prime ideal Q which contains I. Thus we get $P=P\cap Q$, i. e. the factorization of P is not unique.

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[5]) The factorization in case $n>1$ is called irredundant if $P_1\cap\cdots\cap P_{i-1}\cap P_{i+1}\cap\cap\cdots\cap P_n\supset I$ for all i.

(3) *implies* (1). Let $b<c<a$, and suppose that c has no relative complement in the interval $[b, a]$. Let us consider the dual ideal D, formed by the elements d, satisfying $d \cup c \geqq a$, and the dual ideal[6]) $E=(c] \cup D$. Obviously $b \notin E$, for in the contrary case $b=c \cap d$ (see footnote 6) for some $d \in D$, that is, d is a relative complement of c in $[b, a]$. So Lemma I may be used (for $I=(b]$ and E), that is, there exists a prime ideal P, such that P is disjoint to E and $b \in P$. At last we consider the ideal $(c] \cup P$. It is clear that a is not an element of $(c] \cup P$, for in the contrary case $a=c \cup p$ for some $p \in P$, hence by the definition of $D, p \in D$, contrary to the fact that P and E, and consequently, P and D are disjoint. According to Lemma I, there exists a prime ideal Q containing $(c] \cup P$ such that $a \notin Q$. By the definitions, Q properly contains P, that is, the prime ideal P is not maximal.

(1) *implies* (2). Let

$$(*) \qquad I=P_1 \cap \cdots \cap P_n=Q_1 \cap \cdots \cap Q_k$$

be two factorizations of the ideal I of the relatively complemented distributive lattice L. At first we show that these factorizations are irredundant. Indeed, if $I=P_2 \cap \cdots \cap P_n$, then by the distributivity of the lattice of all ideals of a distributive lattice (see [1], p. 141), $P_1=P_1 \cup I=(P_1 \cup P_2) \cap \cdots \cap (P_1 \cup P_n)$, but every prime ideal in the lattice of all prime ideals of L is meet-irreducible (for, if P is a prime ideal and $P=I \cap J$, $P \neq I$, $P \neq J$, then let us choose an $x \in I-P$ and a $y \in J-P$; obviously, $x \cap y \in I \cap J=P$, a contradiction), hence for some i, $P_1 \cup P_i=P_1$, i. e. the prime ideal P_i is not maximal. By Theorem IV the two factorizations of I are the same, for in every relatively complemented distributive lattice all prime ideals are maximal (see [1], p. 160), that is, (1) implies (2).

Conditions (3) and (3′) are dual to each other, condition (1) is selfdual, consequently, (1) is equivalent to (3′) too, completing the proof.

T h e o r e m VI (Lemma 4.3 of [5]). *If b covers a, then there exists at most one prime ideal P such that $a \in P$ and $b \notin P$. Accordingly, if n is the length of the shortest connected chain of the interval $[a, b]$, then there exist at most n prime ideals which contain a but not b.*[7])

P r o o f. Let us suppose that P and Q are prime ideals, $a \in P, Q$ and $b \notin P, Q$. If $P \not\supseteq Q$, then let us choose a $y \in Q-P$. Let $x=y \cup a \in Q-P$, so that $x \notin P$. Obviously, $b \cap x=a$, but $a \in P$, a contradiction.

[6]) The join of the dual ideals A and B is the dual ideal C generated by A and B. In distributive lattices (see [1], p. 141) any $c \in C$ is of the form $c=a \cap b$ $(a \in A, b \in B)$. Hence if $b \in [c) \cup D$, then $b=\bar{c} \cap d$ $(\bar{c} \geqq c, d \in D)$ and if, moreover, $b \leqq c$, then $b=c \cap b= =c \cap \bar{c} \cap d=c \cap d$.

[7]) This is a somewhat sharpened form of Lemma 4.3 of [5]. We were unable to find in [5] the proof of the following Corollary.

Corollary. *In the lattice L, let n and m be the (finite) lengths of the shortest and the longest connected chains, respectively. Then L does not contain more than n prime ideals. L is distributive if and only if it contains m prime ideals.*

Proof. The first statement of the Corollary follows evidently from Theorem VI.

If L is a distributive lattice, and $1=a_0\succ a_1\succ\cdots\succ a_m=0$ ($x\succ y$ means that x covers y) is a (maximal) chain of length m, then, by Lemma I, for all i there exists a prime ideal which contains a_i, but not a_{i-1}. Consequently, in L there exist at least m prime ideals.

Conversely, let us suppose that L contains m prime ideals. These obviously separate (by Theorem VI) the chain $1=a_0\succ a_1\succ\cdots\succ a_m=0$ (i. e. for $i=0,\ldots,m-1$ there exists a prime ideal P_{i+1} such that $a_i\notin P_{i+1}$, $a_{i+1}\in P_{i+1}$). By Theorem II, it is sufficient to prove that these separate all pairs of elements $y<x$. Let $1=b_0\succ b_1\succ\cdots\succ b_n=0$ be a maximal chain which is a refinement of $1\geqq x>y\geqq 0$. Owing to Theorem VI in L there is at most n prime ideals. We have supposed the existence of precisely m prime ideals and we know that $n\leqq m$; it follows $n=m$ and the fact that every pair b_i, b_{i+1} ($i=0,1,\ldots,n-1$) is separated by some P_j. Thus, obviously, some P_j separates x and y too.

We have shown (Corollary 1 of Theorem I) that any ideal of a distributive lattice is the product of its prime ideal divisors. The following problem arises: in what lattices is every factorization unique?

Theorem VII (Theorem 4.3 of [5]). *The following statements concerning a lattice L are equivalent:*

(1) *every ideal of L is decomposed uniquely into the product of prime ideals;*

(2) *every principal ideal of L is decomposed uniquely into the product of prime ideals;*

(3) $\mathfrak{L}$ *is a relatively complemented distributive lattice;*

(4) *L is a relatively complemented distributive lattice in which every closed interval has a finite length.*

Proof. By Theorem II, any one of the conditions (1)—(4) implies the distributivity of the lattice L, hence it may be supposed without loss of generality that L is distributive.

(2) *implies* (1). Let us suppose, in contradiction to (1), that there exists an ideal I which is factorizable into prime divisors in two ways: $I=\wedge P_\alpha=$ $=\wedge Q_\beta$. Let a be an element of I and let us consider the (unique) factor-

ization of $(a] = \wedge R_\gamma$. Obviously $\wedge R_\gamma \frown \wedge P_\alpha$ and $\wedge R_\gamma \frown \wedge Q_\beta$ are (after omitting the R_γ equal to some P_α and Q_β, respectively) two different factorizations of $(a]$.

(1) *implies* (3). We consider ideals I and J such that $I \supset J$. Let $I = \wedge P_\alpha$ and $J = \wedge P_\alpha \frown \wedge P_\beta$ be the factorizations of I and J where all ideals P_β are different from all P_α. Consequently, $P_\beta \not\supseteq I$ for all P_β. We assert that $(\wedge P_\alpha) \smile (\wedge P_\beta) = L$. Indeed if $(\wedge P_\alpha) \smile (\wedge P_\beta) \neq L$, then there exists a prime ideal P which contains $(\wedge P_\alpha) \smile (\wedge P_\beta)$ and so $P \supseteq I$, therefore P is equal to some P_α, so P_α may be omitted from the factorization of J, in contradiction to (1). Hence every interval of the type $[J, L]$ is complemented, therefore $\mathfrak{L}$, which is distributive, is relatively complemented.

(3) *implies* (4). Let $b < a$ $(a, b \in L)$ and let I be an ideal such that $(a] \supset I \supset (b]$. From (3), I has a relative complement in the interval $[(b], (a]]$. Hence by Lemma II, I is a principal ideal, i. e. (4) is indeed proved to be true in L, for the interval $[b, a]$ of L — obviously — is a finite Boolean algebra in which every ideal is principal.[8])

(4) *implies* (2). Let $(a]$ be a principal ideal which has two different factorizations $(a] = \wedge P_\alpha = \wedge Q_\beta$. We choose an element $b > a$. Obviously, b is not element of all P_α and Q_β, e. g. let $b \notin P_1$ and $b \notin Q_1$. Combining condition (4) with Theorem V, we get that in L every prime ideal is maximal, hence $(b] \smile P_1 = (b] \smile Q_1 = L$. Since in a distributive lattice the relative complement is unique, we conclude $(b] \frown P_1 \neq (b] \frown Q_1$, furthermore $(b] \frown P_\alpha$ is a prime ideal in $(b]$. If we consider those elements of $\wedge((b] \frown P_\alpha)$ and $\wedge((b] \frown Q_\beta)$ which are in the interval $[a, b]$, we get obviously two different factorizations of the element a in the finite Boolean algebra $[a, b]$; this is clearly a contradiction.

6. Ideals and congruence relations.

According to Theorem II every ideal of a distributive lattice is a kernel of a suitable homomorphism. In general, it is possible that there exist more than one homomorphisms with the same kernel. G. BIRKHOFF proposed the following problem (see [1], p. 161):

Find necessary and sufficient conditions, in order that the correspondence between the congruence relations and ideals of a lattice be one-one.

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[8]) See [1], p. 161, Ex. 3. We can prove it in the following way: If in the Boolean algebra B every ideal is principal, then every maximal ideal is also principal, that is, B is dually atomic, and hence atomic. The ideal I generated by the atoms of B, contains exactly those elements of B which are finite joins of the atoms of B, and the zero of B. I is a principal ideal, and the generating element x is a finite join of atoms. Obviously x is the greatest element of B and so B is finite.

This problem is answered in the following

Theorem VIII (Theorem 7.2 of [5]). *The congruence relations and the ideals of a lattice L correspond one-to-one if and only if L is a relatively complemented distributive lattice with* 0.

Proof. *Necessity.* The trivial (identical) homomorphism ought to have a kernel, hence 0 exists. The necessity of the distributivity is assured by Theorem II (condition (3)). At last, in order to verify the necessity of relative complementedness (we may already suppose that L is distributive), by Theorem V it is enough to prove that every prime ideal is maximal. But, in the contrary case there exist in L two prime ideals P and Q such that $P \subset Q$. It is evident that there exists a homomorphism with the kernel P, such that the homomorphic image is isomorphic to the lattice of two elements. On the other hand let $A_1 = P$, $A_2 = Q - P$, $A_3 = L - Q$. We define the relation Θ: $x \equiv y(\Theta)$ if and only if x and y are in the same A_i $(i = 1, 2, 3)$. It is easy to verify that Θ is a congruence relation; the homomorphism induced by Θ has the kernel P, and the homomorphic image is isomorphic to the chain of three elements. Consequently, P is the kernel of more than one homomorphism.

Sufficiency. This follows from Theorem II and from the evident fact that in a relatively complemented lattice with zero element every homomorphism is completely determined by its kernel ([1], p. 23).

9. Maximal extension of sublattices.

Generalizing a theorem of K. TAKEUCHI [6], J. HASHIMOTO proves that any sublattice of a relatively complemented distributive lattice may be extended to a proper, maximal one. In proving it he uses a lemma and by its aid he proves Theorem 9.1; both the lemma and the theorem are proved by making use of some topologies. The other parts of the proof have purely lattice-theoretical character. For this reason now we prove only Theorem 9.1 (we need not use the lemma).

Theorem IX (Theorem 9.1 of [5]). *Let a be an element of a distributive lattice L, which is neither* 0 *nor* 1, *and let S be a sublattice of L which does not contain a. Then there exist a prime ideal P and a dual prime ideal Q, such that (denoting by $P+Q$ the set-theoretical join of P and Q) $P+Q \supseteq S$ and $a \notin P+Q$.*

Proof. Let us consider (if it exists) the ideal I generated by those elements of S which are less than a. Obviously $a \notin I$, consequently, applying Lemma I for I and $[a)$, there exists a prime ideal P containing I but not

a. Now we consider the dual ideal D, generated by those elements of S which are not in P (if such elements exist). We prove that $a \notin D$. Indeed, $a \in D$ is equivalent to $a \geqq s \frown t$, where $s, t \in S$, but $s, t \notin P$. Since $a \notin S$, $a = s \frown t$ is impossible. Furthermore $a > s \frown t$ implies, by the definition of P, that $s \frown t \in P$, in contradiction to the prime property of P. Using again Lemma I, we may construct a dual prime ideal Q containing D but not a. P and Q fulfil the requirements.

If I is empty, then let P be any prime ideal not containing a. If D is empty then $P \supseteq S$, therefore Q may be an arbitrary dual prime ideal not containing a.

Added in proof. It escaped our attention that in his paper [5] J. HASHIMOTO also proves the following very interesting theorem (Theorem 8.5 of [5]) of purely lattice-theoretical character:

To any distributive lattice L there exists a generalized Boolean algebra B having the properties:

1. the lattice of all congruence relations of L is isomorphic to the lattice of all congruence relations of B;
2. L is a sublattice of B;
3. if the interval $[a, b]$ of L is of finite length, then $[a, b]$ has the same length as an interval of B.

HASHIMOTO devotes to the proof of this theorem an entire section in which he constructs B from L in a rather complicated topological way.

Recently we have succeeded in finding two simple proofs. One of these is an easy consequence of a construction of MAC NEILLE (Lattices and Boolean rings, *Bull. Amer. Math. Soc.*, **45** (1939), 453—455), while the other is based on the examination of the $\uparrow$-inaccessible elements of the lattice of all congruence relations of a lattice and uses some results of [3]. The second proof is also capable of some generalization.

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