On complete semi-groups.

By RICHARD WIEGANDT in Orosháza (Hungary).

§ 1.

By an (algebraic) structure we shall mean in the following a group, a ring or a semi-group. A structure will be called a T-structure if it has some specified additional property T.

It is a well-known fact that the Schreierian extensions of a group or a ring are the groups or rings, in which the given group or ring is a normal subgroup or an ideal, respectively. RÉDEI [3] treated the Schreierian extension theory of semi-groups with identity;[1]) the Schreierian extensions of a semi-group are the semi-groups in which it is a "left-normal semi-group" (see below).

Definition. A T-structure S is called *complete with respect to the property* T (shortly: T-complete) if it is a direct component (i. e. direct factor or direct summand) in every T-structure which is a Schreierian extension of S.

Examples of complete structures are the complete groups among the groups (each of their automorphisms is inner, their center consists of the identity only), the complete Abelian groups among the Abelian groups (for every element a and positive integer n there exists an element x such that $nx=a$), and the rings with identity among the rings. In these examples the property T means group, Abelian group, or ring, respectively (BAER [1], [2], RÉDEI [4]).

In this paper our main purpose is to characterize the complete regular semi-groups[2]) with identity; finally we make some remarks on groups and Abelian groups, which are complete with respect to certain properties.

[1]) A semi-group is a structure in which an associative multiplication is defined. The identity will be denoted always by e.

[2]) A semi-group is regular, when $xz=yz$ or $zx=zy$ implies $x=y$ for every element x, y, z.

§ 2.

Consider the case when the property T means "regular semi-group with identity". First we need some preparatory remarks.

Let F denote a semi-group with the identity e (F is not necessarily regular). According to RÉDEI [3], a sub-semi-group N of F is called left-normal, if F has a compatible classification of the form

$$(1) \qquad a_1N,\ a_2N, \ldots \qquad (a_i \in F,\ a_1 = e)$$

and the products a_iN are without repetition.

Similarly we can define the right-normal semi-group. If N is at the same time left-normal and right-normal in F, then N is called a normal sub-semi-group of F.

Let us consider an example for a left-normal semi-group. Consider the semi-group which is generated by the elements e, a, b (e is the identity) and defined by the relation $ab = b$. It is easy to see that in this semi-group the elements e, a^n ($n \geqq 1$ integer) form a left-normal semi-group, but this left-normal semi-group is not right-normal.

Let b_i ($\in a_iN$) denote an arbitrary element of the class a_iN; then $b_iN \subseteqq a_iN$ ($i = 1, 2, \ldots$), and the equality sign is valid obviously in every case if and only if N is a group.

N contains the identity of F. Otherwise we should have, according to the previous fact, $N = eN \subseteqq a_kN$ ($a_k \neq e$) for some $k \neq 1$, what is impossible.

So F and N have a common identity, further a_k ($k = 2, 3, \ldots$) is contained in the class a_kN, but in general it is not possible to replace a_k by an arbitrary element of the class a_kN.

The classification (1) is determined uniquely by the semi-group N. Consider namely beside (1) an other left-normal classification

$$(2) \qquad b_1N,\ b_2N, \ldots \qquad (b_1 = e)$$

of F. Every a_i belongs to a fixed b_kN, and b_k to a fixed a_lN. Since (2) is compatible, so $a_iN \subseteqq b_kN$; from (1) follows $b_kN \subseteqq a_lN$. Hence $a_iN \subseteqq a_lN$, $a_iN = a_lN$ and $a_iN = b_kN$ follows proving the statement.

Lemma. *If N is normal in the semi-group F with identity, then the left-classes are identical with the right-classes. If a ($\in F$) has an inverse in F then the class of a can be written in the form aN, and we have $aN = Na$.*

Proof. Let a_kN be an arbitrary left-class and let a_k belong to the right-class Nb_l. Since the classification is compatible, so $a_k \equiv b_l$. Multiplying from the right with an arbitrary element r ($\in N$), we get

$$a_kr \equiv b_lr \equiv b_le \equiv b_l,$$

thus $a_k N \subseteq N b_l$. Likewise $N b_l \subseteq a_n N$; from these $k = n$ and $a_k N = N b_l$ follows, which proves the first statement.

Let the element a ($\in F$) have an inverse in F, and let a belong to the class $a_k N$. Then

$$aN \subseteq a_k N.$$

Multiplying from the left with a^{-1}, we get

$$N \subseteq a^{-1} a_k N.$$

Consequently in both relations the equality sign is valid and so the class of a is aN.

The third statement follows from the preceding statements.

Let the property T mean that the structure is a regular semi-group with identity. Such a semi-group is called complete by the above definition, if F is a direct factor of every regular semi-group with identity, which contains it as a left-normal semi-group.

Now we prove the following

Theorem. *The regular semi-group F with identity is complete if and only if its automorphisms are all inner automorphisms, and its center consists of the identity.*

Remark. If in particular F is a group, then the theorem reduces to a known theorem of BAER [1] for groups.

Proof. The proof is a modification of BAER's [1] proof.

Assume that F is complete, and let α be an arbitrary automorphism of F. Consider the factor-free Schreierian extension of F with an infinite[3]) cyclic group: $\bar{B} \approx I \circ F$ (I is the additive group of the integers). The elements of $\bar{B}$ are the pairs (i, f) ($i \in I, f \in F$) in which the multiplication is defined by the following rule:

$$(i, f)(j, g) = (i + j, f^{\alpha^j} g)$$

(α^j is the j-th power of the automorphisms α).

By theorem 1 of RÉDEI [3] $\bar{B}$ is a semi-group, and $\bar{B}$ is obviously regular too. It is clear that $(0, e)$ is the identity of $\bar{B}$. In $\bar{B}$ the elements $(0, f)$ form a left-normal semi-group $\bar{F}$, which is isomorphic to F. Embed F in the usual way into $\bar{B}$; further denote the element $(1, e)$ by t, and denote the so formed semi-group by B. t has an inverse: $t^{-1} = (-1, e)$. Since

$$(i, f) = (1, e)^i (0, f),$$

the elements of B are of the form $t^i f$. Since

$$(1, e)(0, f)(-1, e) = (0, f^{\alpha}),$$

[3]) If the order of the automorphism α is a finite number n, we may take instead of the infinite cyclic group, the cyclic group of order n.

the following relation holds in B:

$$tft^{-1}=f^{\alpha} \qquad (f\in F).$$

Thus the automorphism α of F is induced by the transformation of the element t $(\in B)$. Since F is left-normal in B, therefore F is by the hypothesis a direct factor in B. Hence there exists an endomorphism β of B with the following properties:

$$B^{\beta}=F, \qquad f^{\beta}=f \qquad (f\in F).$$

In particular $s=t^{\beta}\in F$, and

i) there exists an inverse of s in F,

ii) for every element f in F

$$sfs^{-1}=t^{\beta}f(t^{\beta})^{-1}=(tft^{-1})^{\beta}=f^{\alpha\beta}=f^{\alpha}.$$

Thus the automorphism α is induced by the element s of F, thus every automorphism of F is inner.

Let z be an arbitrary element in the center of F. Denote by I' the additive semi-group of the non-negative integers, and consider the direct sum $J=I'+I'$. Consider the endomorphism-free Schreierian extension of F with J: $Z^{*}\approx J\circ F$. The elements of Z^{*} are of the form $((i,j),f)$ $((i,j)\in J, f\in F)$, and the multiplication is defined as follows:

$$((i,j),f)((k,l),g)=((i+k,j+l),fgz^{jk}).$$

By theorem 1 of RÉDEI [3] Z^{*} is a semi-group, further Z^{*} is clearly regular. Obviously $((0,0),e)$ is the identity of Z^{*}. In Z^{*} the elements $((0,0),f)$ form a left-normal semi-group F^{*} which is isomorphic to F. Embed F into Z^{*} and denote the elements $((0,1),e)$, $((1,0),e)$ by x and by y, respectively. Denote the so formed semi-group by Z. It is easy to see that the following relations hold in Z:

$$xy=yxz,\ xf=fx,\ yf=fy \qquad (f\in F).$$

Since F is left-normal in Z, so F is by the hypothesis a direct factor in Z. Hence there exists an endomorphism γ of Z with the following properties:

$$Z^{\gamma}=F, \qquad f^{\gamma}=f \qquad (f\in F).$$

If $f\in F$ then

$$x^{\gamma}f=x^{\gamma}f^{\gamma}=(xf)^{\gamma}=(fx)^{\gamma}=fx^{\gamma},$$

which proves that x^{γ} belongs to the center of F. Analogously, y^{γ} belongs to the center of F. Consequently

$$y^{\gamma}x^{\gamma}=x^{\gamma}y^{\gamma}=(xy)^{\gamma}=(yxz)^{\gamma}=y^{\gamma}x^{\gamma}z^{\gamma}.$$

Since F is regular, we have $z=e$. Hence we have shown that the identity is the only element in the center of F; and so we have proved the necessity of the theorem.

Assume conversely that every automorphism of F is inner, and its center consists of the indentity only, further F is left-normal in the regular semi-group D with identity. Denote by C the centralizer of F in D (which consists of all those elements in D, which commute with every element in F).

We show that $CF=D$. Otherwise there would exist an element w $(\in D)$ which does not belong to any class cF $(c\in C)$. Let w belong to the class w_0F $(w_0\notin C)$. It may be assumed that w_0 has an inverse; otherwise we should consider the semi-group obtained by adjoining to D an element w_0^{-1} subjected to the following relation: $w_0^{-1}w_0=e$; since D is regular, the obtained semi-group is an extension of F in which F is also left-normal. w_0 induces an automorphism of F, which contradicts the condition that every automorphism of F is inner. Consequently $D=CF$. Since $C\cap F=e$ according to the hypothesis, therefore D is the direct product of F and C. Hence F is direct factor in D, and this completes the proof.

§ 3.

Intermediate concepts between those of general groups and Abelian groups are the concepts of soluble groups and nilpotent groups. Consider the complete soluble and complete nilpotent groups. It is easy to see by the proof of the theorem that the center of a complete soluble or complete nilpotent (or other complete not Abelian) group must be the identity. On the other hand every soluble (and so every nilpotent) group has non-trivial center. So every complete soluble and complete nilpotent group must be the identity.

Every finitely generated complete Abelian group is the identity. They are namely the direct products of cyclic groups; but the cyclic groups are not direct factors in the containing cyclic groups.

The author is grateful to Professor L. Rédei who kindly helped him with the preparation of this paper.

Bibliography.

[1] Baer, R, Absolute retracts in group theory, *Bull. Amer. Math. Soc.*, **52** (1946), 501—506.

[2] Baer, R. Abelian groups that are direct summands of every containing Abelian group, *Bull. Amer. Math. Soc.*, **46** (1940), 800—806.

[3] Rédei, L., Die Verallgemeinerung der Schreierschen Erweiterungstheorie, *Acta Sci. Math.*, **14** (1952), 252—273.

[4] Rédei, L., Die Holomorphentheorie für Gruppen und Ringe, *Acta Math. Acad. Sci. Hung.*, **5** (1954), 169—195.

(Received September 29, 1957.)