On almost orthogonal operators in L^p-spaces.

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Let $K_1(x), \ldots, K_N(x)$ be integrable functions defined on the n-dimensional Euclidean space $E^n = \{x\}$, $x = \{\xi_1, \ldots, \xi_n\}$ (we shall also identify the point x with the vector Ox and use vector notations such as $x - y$, $|x| = (\xi_1^2 + \cdots + \xi_n^2)^{1/2}$), and let

(1) $$T_i f = f * K_i = \int_{E^n} f(y)\, K_i(x-y)\, dy,$$

(2) $$K(x) = K_1(x) + \cdots + K_N(x),$$

(3) $$Tf = f * K = \sum T_i f = \int_{E^n} f(y)\, K(x-y)\, dy,$$

(4) $$K_{ij}(x) = K_i * K_{i+j}(x) \qquad (1 \leqq i \leqq i+j \leqq N),$$

(5) $$\|f\|_p = \left\{ \int_{E^n} |f(x)|^p\, dx \right\}^{1/p} \qquad (dx = d\xi_1 \ldots d\xi_n).$$

In a previous paper [1] one of the authors proved the following

Theorem A. *If the kernels K_{ij} satisfy the conditions*

(6) $$\|K_{ij}\|_1 \leqq c \cdot \varepsilon^j \qquad (1 \leqq i \leqq i+j \leqq N),$$

where $0 \leqq \varepsilon < 1$, and if $f \in L^2(E^n)$, then

(7) $$\|f * K\|_2 \leqq c_1 \|f\|_2, \qquad c_1 = c_1(\varepsilon, c),$$

where the constant c_1 depends on ε and c only, and not on N.

(Since T_i are operators on L^2 with $\|T_i\| \leqq \|K_i\|_1$, and since (6) implies $\|T_i T_{i+j}\| \leqq c \cdot \varepsilon^j$, we say that the T_i are "almost orthogonal" operators on L^2). B. Sz.-Nagy [2] gave a very simple proof of Theorem A (and of a more general theorem) by reducing it to the following numerical lemma:

Lemma A. *For any sum $s = v_1 + \cdots + v_N$ of real numbers with $|v_i v_{i+j}| \leqq \varepsilon^j$ $(1 \leqq i \leqq i+j \leqq N)$, it is true that $s \leqq c(\varepsilon)$.*

Here we give the following generalizations of Theorem A to L^p-spaces and to subspaces of E^n:

T h e o r e m B. *Let* $0 \leqq \gamma < n =$ *dimension of* E^n. *Let the kernels* K_{ij} *satisfy the conditions*

(8) $$\int_{E^n} |K_{ij}(x+h) - K_{ij}(x)|\, dx \leqq c \cdot \varepsilon^j \cdot |h|^{2\gamma} \qquad (1 \leqq i \leqq i+j \leqq N),$$

for an ε, $0 \leqq \varepsilon < 1$, *and for every* $h \in E^n$, *if* $p = 2n/(n+\gamma)$, $1/p + 1/q = 1$, *then*

(9) $$\|f * K\|_q \leqq c_1 \cdot \|f\|_p$$

holds for every $f \in L^p(E^n)$, *where* c_1 *depends on* ε, γ *and* c *only.*

(Theorem B reduces to Theorem A for $\gamma = 0$ and $p = 2$.)

T h e o r e m C. *Let* $0 \leqq \gamma < n$, $m < n < m + 2\gamma$, $E^m \subset E^n$. *Let in formula* (3) y *vary in* E^n *and* x *in* E^m, *so that* f *and* K *are defined in* E^n, *while* $F(x) = f * K$ *is considered as a function defined in* E^m. *Then, if the kernels* K_{ij} *satisfy conditions* (8), *we have for every* $f \in L^p(E^n)$

(10) $$\|F\|_q^{(m)} = \|f * K\|_q^{(m)} \leqq c_1 \|f\|_p^{(n)}, \qquad c_1 = c_1(\varepsilon, \gamma, c),$$

where $p = (n+m)/(m+\gamma)$, $1/p + 1/q = 1$, *and*

$$\|F\|_q^{(m)} = \left\{ \int_{E^m} |F(x)|^q\, dx \right\}^{1/q}.$$

T h e o r e m D. *Let* $0 \leqq \gamma < m$, $m < n < m + 2\gamma$, $E^m \subset E^n$. *Let in formula* (3) y *vary in* E^m *and* x *in* E^n, *so that* f *is defined on* E^m, *while* K *and* $F = f * K$ *are defined on* E^n. *Then, if the kernels* K_{ij} *satisfy conditions* (8), *we have*

(11) $$\|F\|_q^{(n)} = \|f * K\|_q^{(n)} \leqq c_1 \|f\|_p^{(m)}, \quad c_1 = c_1(\varepsilon, \gamma, c),$$

for every $f \in L^p(E^m)$, *where* $p = (n+m)/(m+\gamma)$, $1/p + 1/q = 1$.

Theorems B, C, D are easy consequences of Lemma A and the following lemmas:

L e m m a B. *Let* $0 \leqq \gamma < n$. *Let* $f(x)$, $K(x)$ *and* $F(x) = f * K$ *be defined on* E^n, *and assume that*

(12) $$|u|^\gamma \cdot |\hat{K}(u)| \leqq c$$

holds for all $u \in E^n$, *where* $\hat{K}$ *is the Fourier transform of* K. *Then*

(13) $$\|f * K\|_q \leqq c_1 \|f\|_p, \quad c_1 = c_1(\gamma, c),$$

with $p = 2n/(n+\gamma)$, $1/p + 1/q = 1$.

Lemma C. *Let* $0 \leqq \gamma < n$, $m < n < m+2\gamma$, $E^m \subset E^n$. *Let* f, K *be defined on* E^n *while* $F = f*K$ *is considered as defined on* E^m. *Then, if the kernel* K *satisfies condition* (12), *it is true that*

$$\|f*K\|_q^{(m)} \leqq c_1 \|f\|_p^{(n)}, \quad c_1 = c_1(\gamma, c), \tag{14}$$

with $p = (m+n)/(m+\gamma)$, $1/p + 1/q = 1$.

Lemma D. *Let* $0 \leqq \gamma < m$, $m < n < m+2\gamma$, $E^m \subset E^n$. *Let* f *be defined on* E^m, *while* K *and* $F = f*K$ *are defined on* E^n, *and let the kernel* K *satisfy condition* (12) *for all* $u \in E^n$. *Then*

$$\|f*K\|_q^{(n)} \leqq c_1 \|f\|_p^{(m)}, \tag{15}$$

with $p = (n+m)/(m+\gamma)$, $1/p + 1/q = 1$.

Proof of Lemma B. For every function $g \in L^p(E^n)$ and $1/p + 1/q = 1$, $1 < p \leqq 2$, we have the following classical inequalities of Hausdorff—Young and Hardy—Littlewood—Paley ([3], Chap. 9):

$$\left\{ \int_{E^n} |g(x)|^q \, dx \right\}^{1/q} \leqq \left\{ \int_{E^n} |\hat{g}(u)|^p \, du \right\}^{1/p}, \qquad \int_{E^n} |\hat{g}(u)|^p |u|^{n(p-2)} \, du \leqq c_p \int_{E^n} |g(x)|^p \, dx,$$

where $\hat{g}$ is the Fourier transform of g. Using these inequalities and hypothesis (12), and taking in account that $p = 2n/(n+\gamma)$, $p\gamma = n(2-p)$, we obtain

$$\|f*K\|_q = \|F\|_q \leqq \left\{ \int_{E^n} |\hat{F}(u)|^p \, du \right\}^{1/p} = \left\{ \int_{E^n} |\hat{f}(u)|^p |\hat{K}(u)|^p \, du \right\}^{1/p} \leqq$$

$$\leqq c \left\{ \int_{E^n} |\hat{f}(u)|^p |u|^{-p\gamma} \, du \right\}^{1/p} = c \left\{ \int_{E^n} |\hat{f}(u)|^p |u|^{n(p-2)} \, du \right\}^{1/p} \leqq c_1 \|f\|_p .$$

Proof of Lemma C. Now f and K are defined on E^n, and $f*K$ on E^m, $m < n$. Let $E^m = \{t\} = \{x\} = \{v\}$, $E^{n-m} = \{z\} = \{w\}$, $E^n = E^m \times E^{n-m} = \{y\} = \{(t, z)\} = \{u\} = \{(v, w)\}$, and let $f(y) = f(t, z)$. Then

$$F(x) = \int_{E^m} dt \int_{E^{n-m}} f(t, z) \, K(x-t, -z) \, dz \tag{16}$$

and

$$\hat{F}(v) = \int_{E^m} F(x) e^{i(x,v)} dx = \int_{E^m} e^{i(x,v)} dx \int_{E^m} dt \int_{E^{n-m}} f(t, z) \, K(x-t, -z) \, dz =$$

$$= \int_{E^{n-m}} dz \int_{E^m} f_z(t) \, dt \int_{E^m} K(x, -z) \, e^{i(x,v)} e^{i(t,v)} dx = \int_{E^{n-m}} \hat{f}_z(v) \, \hat{K}_{-z}(v) \, dz, \tag{17}$$

where $f_z(v)$ is the Fourier transform of $f_z(t)$, considered as a function of t with fixed z. Hence

$$(17a)\qquad |\hat{F}(v)| \leqq \left\{ \int_{E^{n-m}} |\hat{f}_z(v)|^p \, dz \right\}^{1/p} \left\{ \int_{E^{n-m}} |\hat{K}_z(v)|^q \, dz \right\}^{1/q}.$$

Here $\varphi_v(z) = \hat{K}_z(v)$ is the Fourier transform of $K_z(t)$. If we take now the Fourier transform $\hat{\varphi}$ of $\varphi_v(z)$, considered as a function of z, we obtain

$$\hat{\varphi}_v(w) = \int_{E^{n-m}} \hat{K}_z(v) e^{i(z,w)} dz = \int_{E^{n-m}} \int_{E^m} K(t,z) e^{i(t,v)} e^{i(z,w)} dt\, dz =$$

$$= \int_{E^n} K(y) e^{i(y,u)} dy = \hat{K}(u) = \hat{K}(v, w).$$

Hence, applying Hausdorff—Young inequality to $\varphi_v(z)$ and taking into account that $|\hat{K}(u)| \leqq c|u|^{-\gamma} = c(|v|^2 + |w|^2)^{-\gamma/2}$, we shall have

$$\left\{ \int_{E^{n-m}} |\hat{K}_z(v)|^q dz \right\}^{1/q} \leqq \left\{ \int_{E^{n-m}} |\hat{\varphi}_v(w)|^p dw \right\}^{1/p} = \left\{ \int_{E^{n-m}} |\hat{K}(v,w)|^p dw \right\}^{1/p} \leqq$$

$$(18)\qquad \leqq c \left\{ \int_{E^{n-m}} (|v|^2 + |w|^2)^{-p\gamma/2} dw \right\}^{1/p} = c \left\{ |v|^{n-m-p\gamma} \int_{E^{n-m}} (1 + |w|^2)^{-p\gamma/2} dw \right\}^{1/p}.$$

Since $p = (m+n)/(m+\gamma)$ and $n < m + 2\gamma$, we have $p < 2$ and $p\gamma = n + m - pm > n - m$, and the last integral of (18) is finite, and since $n - m - p\gamma = n(p-2)$, we obtain from (17a) and (18), that

$$|\hat{F}(v)| \leqq c_1 |v|^{n(p-2)/p} \left\{ \int_{E^{n-m}} |\hat{f}_z(v)|^p dz \right\}^{1/p}.$$

Hence, using the inequalities of Hausdorff—Young and of Hardy—Littlewood—Paley, we obtain

$$\|F\|_q^{(m)} \leqq \left\{ \int_{E^m} |\hat{F}(v)|^p dv \right\}^{1/p} \leqq c_1 \left\{ \int_{E^m} \left[|v|^{n(p-2)} \int_{E^{n-m}} |\hat{f}_z(v)|^p dz \right] dv \right\}^{1/p} =$$

$$= c_1 \left\{ \int_{E^{n-m}} dz \int_{E^m} |\hat{f}_z(v)|^p |v|^{n(p-2)} dv \right\}^{1/p} \leqq c_2 \left\{ \int_{E^{n-m}} dz \int_{E^m} |f(t,z)|^p dt \right\}^{1/p} = c_2 \|f\|_p^{(n)}.$$

Proof of Lemma D. Let $E^n = E^m \times E^{n-m}$, $E^m = \{t\} = \{y\} = \{v\}$, $E^{n-m} = \{z\} = \{w\}$, $E^n = \{x\} = \{(y,z)\} = \{u\} = \{(v,w)\}$, so that

$$F(x) = F(y,z) = \int_{E^m} f(t)\, K(y-t, z)\, dt$$

and

$$\hat{F}(u) = \hat{F}(v, w) = \int_{E^n} e^{i(x,u)}\, dx \int_{E^m} f(t)\, K(y-t, z)\, dt =$$

(19)
$$= \int_{E^m} e^{i(y,v)}\, dy \int_{E^{n-m}} e^{i(z,w)}\, dz \int_{E^m} f(t)\, K(y-t, z)\, dt = \hat{f}(v)\, \hat{K}(v, w).$$

Using (12) and taking in account that $p\gamma = n+m-pm > n-m$, we have

(20)
$$\int_{E^{n-m}} |\hat{K}(v, w)|^p\, dw \leqq c \int_{E^{n-m}} (|v|^2 + |w|^2)^{-p\gamma/2}\, dw =$$
$$= c \int_{E^{n-m}} |v|^{n-m-p\gamma} (1 + |w|^2)^{-p\gamma/2}\, dw = c_1 |v|^{m(p-2)}.$$

Hence, using (19) and (20), we obtain

$$\|F\|_q^{(n)} \leqq \left\{ \int_{E^n} |\hat{F}(u)|^p\, du \right\}^{1/p} = \left\{ \int_{E^{n-m}} \int_{E^m} |\hat{F}(v, w)|^p\, dv\, dw \right\}^{1/p} =$$
$$= \left\{ \int_{E^m} |\hat{f}(v)|^p\, dv \int_{E^{n-m}} |\hat{K}(v, w)|^p\, dw \right\}^{1/p} \leqq$$
$$\leqq c_1 \left\{ \int_{E^m} |\hat{f}(v)|^p\, |v|^{m(p-2)}\, dv \right\}^{1/p} \leqq c_2 \left\{ \int_{E^m} |f(y)|^p\, dy \right\}^{1/p} = c_2 \|f\|_p^{(m)}.$$

P r o o f o f T h e o r e m s B, C, D. In virtue of lemmas B, C, D it is sufficient to prove that the hypothesis (8) implies condition (12). For any function $g(y)$, $y \in E^n$, we have

$$\hat{g}(u) = \int_{E^n} g(y)\, e^{i(y,u)}\, dy, \quad \hat{g}(u)\, e^{i(h,u)} = \int_{E^n} g(y-h)\, e^{i(y,u)}\, dy,$$
$$|\hat{g}(u)(1 - e^{i(h,u)})| \leqq \int_{E^n} |g(y-h) - g(y)|\, dy.$$

Letting $h = u/|u|^2$, so that $|h| = 1/|u|$, and $g = K_{ij}$, we obtain from (8) that

$$|\hat{K}_{ij}(u)| \leqq c \cdot \varepsilon^j |u|^{-2\gamma}.$$

Since $K_{ij} = K_i * K_{i+j}$, we obtain $|\hat{K}_i(u)|\, |\hat{K}_{i+j}(u)| \leqq c\, \varepsilon^j |u|^{-2\gamma}$, or

(21)
$$(|\hat{K}_i(u)|\, |u|^\gamma) \cdot (|\hat{K}_{i+j}(u)|\, |u|^\gamma) \leqq c\varepsilon^j.$$

Applying Lemma A we obtain from (21) that

$$|\hat{K}(u)|\, |u|^\gamma \leqq \sum |\hat{K}_i(u)|\, |u|^\gamma \leqq c_1(\varepsilon, c).$$

This proves the theorems.

Applications. *a)* Let $K(x)=\sum_{1}^{\infty} K_i(x)$ and assume that all $K_i(x) \geqq 0$ and that (8) holds for all $i, j < \infty$. Then, since $|f*K(x)| \leqq \sum |f| * K_i(x)$, we deduce easily that Theorems B, C, and D apply to the operator $f*K$.

b) Let $y=(y_1, \dots, y_n)$ and consider the operator

$$(22) \qquad H_{\gamma n} f(x) = F(x) = \overbrace{\int_0^{\infty} \cdots \int_0^{\infty}}^{n} f(y) \cdot |x-y|^{\gamma - n} \, dy_1 \cdots dy_n,$$

with $0 < \gamma \leqq n$. Let $K_i(y) = |y|^{\gamma-n}$ if $2^i \leqq |y| < 2^{i+1}$, and zero otherwise, and let $K = \sum_{-\infty}^{\infty} K_i$. Then

$$(22a) \qquad H_{\gamma n} f = F = f * K.$$

It is easy to check that the kernels K_i thus defined satisfy conditions (8), and thus we obtain the following

Corollary. *The inequalities* (9), (10) *and* (11) *are true for the operator* $H_{\gamma n} f$.

For $m=n=1$, the corollary is a special case of a theorem of HARDY—LITTLEWOOD [3], and for $m=n>1$ it is a special case of a theorem due to SOBOLIEFF [4]. For $m<n$ and with E^m, E^n replaced by bounded sets, as well as with q replaced by $s<q$, it was proved by SOBOLIEFF [5], who proposed the full inequality (10) as a problem. The part $m>n$ of the Corollary is probably new. More general and complete results of this kind are given in [6].

Generalizations. 1. The inequality of HAUSDORFF—YOUNG used in the above proofs, is a particular case of the following more general inequality, due to PITT [7]:

$$\|\hat{f}\|_q \leqq \left\{ \int_{E^n} |f(x)|^p |x|^{n\alpha p} dx \right\}^{1/p}, \quad 0 \leqq \alpha < 1 - \frac{1}{p}, \quad q \geqq p, \quad \frac{1}{p} + \frac{1}{q} = 1 - \alpha.$$

Using this inequality in the above proofs, we will obtain that the hypothesis (8) implies the inequality (9) for any (p, q) such that $1/p - 1/q = \gamma/n$. However, since PITT's theorem imposes the restriction $1 < p \leqq 2$, $0 \leqq \alpha < 1 - 1/p$, the proof applies only for p such that $1/2 \leqq p \leqq 1/2 + \gamma/2n$ (for instance, if $\gamma = 0$, the proof applies only for $p=2$). Similiar remarks apply to Theorems C and D. It would be interesting to extend the above proofs also to the values p with $1/p \geqq \frac{1}{2} + \gamma/2n$.

2. In the case $\gamma=0, p=2$, Theorem A remains true (see [1] or [2]) if the operators $T_i f=f*K_i$ are replaced by arbitrary hermitean operators on L^2 (or on a Hilbert space). It would be interesting to obtain similiar generalizations of Theorems B, C, D, in terms of operator theory.

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