On strongly continuous semigroups of spectral operators in Hilbert space.

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It was proved by BÉLA SZ.-NAGY [4] that any uniformly bounded one-parameter group of linear operators $\{T_t\}$ on a Hilbert space $\mathfrak{H}$ is similar to a one-parameter group of unitary operators $\{U_t\}$ (i. e. there exists a regular self-adjoint operator A, such that $T_t = AU_tA^{-1}$ for all $-\infty < t < +\infty$). Using especially this fact, we shall prove the following

Theorem 1. *Any strongly continuous one-parameter semigroup $\{T_t\}$ $(t>0)$ of scalar type operators (in* DUNFORD*'s sense* [2]) *on a Hilbert space $\mathfrak{H}$, having their spectral measures $E_t(\sigma)$ uniformly bounded for $t>0$,*[1]) *is similar to a semigroup $\{N_t\}$ of normal operators, i. e. there exists a regular selfadjoint operator A on $\mathfrak{H}$ such that $T_t = AN_tA^{-1}$ for all $t>0$.*

1. We shall first consider a particular case. Let us call a scalar type operator T *circled* if its spectrum $\sigma(T)$ lies on the unit circle $\{\lambda : |\lambda| = 1\}$. Then the condition of uniform boundedness with respect to t is unnecessary.

Theorem 2. *Any strongly continuous one-parameter semigroup $\{T_t\}$ $(t>0)$ of circled scalar type operators is similar to a semigroup of unitary operators.*

Proof. Since T_t is circled, T_t^{-1} exists for all $t>0$; if we put $T_0 = I$, and $T_t = T_{-t}^{-1}$ for $t<0$, we obtain a one-parameter group of operators $\{T_t\}$. Since $T_tf = T_{1+t}T_1^{-1}f \to T_1T_1^{-1}f = f$ for $t \to 0$ and $f \in \mathfrak{H}$, the one-parameter group $\{T_t\}$ is strongly continuous at $t=0$, thus for all real t. Put $\mu = \sup_{0 \leq t \leq 1} \|T_t\|$; from the STEINHAUS—BANACH theorem it follows that $\mu < +\infty$. Let $[t]$ be the greatest integer $\leq t$; by [2], theorem 7, we have

$$\|T_t\| \leq \|T_{t-[t]}\| \cdot \|T_{[t]}\| \leq \mu \|T_{[t]}\| = \mu \|T_1^{[t]}\| \leq \mu \cdot v(E_1) \sup_{\lambda \in \sigma(T_1)} |\lambda|^{[t]} = \\ = \mu \cdot v(E_1) < +\infty,$$

where $v(E_1)$ is a finite constant depending only on the spectral measure $E_1(\sigma)$ of T_1. Thus, the one-parameter group $\{T_t\}$ is uniformly bounded.

[1]) I. e. there exists a $K < \infty$ such that $\|E_t(\sigma)\| \leq K$ for all $t > 0$ and all Borel set σ on the real axis $\Omega = (-\infty, \infty)$.

By B. Sz.-Nagy's theorem mentioned above, there exists a regular selfadjoint operator A such that $A^{-1}T_tA$ is unitary for all t; hence theorem 2 is proved.

2. In the proof of theorem 1 we need also the "polar decomposition" of a scalar type operator T. Let $\sigma(T)$ be its spectrum, and $E(\sigma)$ the spectral measure of T. Put

(1) $r(\lambda)=|\lambda|,\quad u(\lambda)=\exp(i\arg\lambda)\quad\text{for}\quad \lambda\neq 0,\quad\text{and}\quad u(0)=1.$

Then if

(2) $$R=\int_{\sigma(T)} r(\lambda)E(d\lambda),\qquad U=\int_{\sigma(T)} u(\lambda)E(d\lambda),$$

we have by lemma 6 and theorem 16 of [2] that: (i) U is a *circled* scalar type operator, and R a *positive* scalar type operator (i. e. with the spectrum on the positive real semi-axis), (ii) U and R commute with T, and

(3) $$T=RU=UR$$

(4) $$UE(\{0\})=E(\{0\})U=E(\{0\}).$$

Lemma. *R and U are uniquely determined by* (i), (ii), (3), *and* (4) *(R being uniquely determined already by the first three conditions).*

Proof. Let $T=U_1R_1$ be another decomposition of T with the properties (i), (ii) and (3). Since U_1 and R_1 commute with T, by theorem 5 of [2] they commute also with U and R. By an obvious extension of a theorem of J. Wermer ([5], theorem 1) from the case of two commuting spectral measures to the case of four, there is a regular selfadjoint operator A such that $R^0=A^{-1}RA$, $R_1^0=A^{-1}R_1A$, $U^0=A^{-1}UA$ and $U_1^0=A^{-1}U_1A$ are all normal; then U^0, U_1^0 are unitary, and R^0, R_1^0 positive selfadjoint operators; hence from

$$(R^0)^2=R^0U^0(U^0)^*R^0=A^{-1}TA(A^{-1}TA)^*=R_1^0U_1^0(U_1^0)^*R_1^0=(R_1^0)^2,$$

it results that $R^0=R_1^0$. Thus $R=R_1$. To prove the uniqueness of U under the additional condition (4), remark that for all $f\in\mathfrak{H}$ we have

$$T(U-U_1)f=UR(U-U_1)f=U(T-T)f=0,$$

so that $E(\{0\})(U-U_1)f=(U-U_1)f$. This relation gives $U[I-E(\{0\})]=$ $=U_1[I-E(\{0\})]$; so that if U_1 satisfies also (4), we obtain $U=U_1$, and the lemma is proved.

3. We can now pass to the proof of theorem 1. Let $\{T_t\}$ $(t>0)$ be a strongly continuous semigroup of scalar type operators. Put

$$R_t=\int_{\sigma(T_t)} r(\lambda)E_t(d\lambda),\qquad U_t=\int_{\sigma(T_t)} u(\lambda)E_t(d\lambda),\qquad V_t=\int_{\sigma(T_t)} \overline{u(\lambda)}\,E_t(d\lambda),$$

where $E_t(\sigma)$ is the spectral measure of T_t. Remark that $V_t = U_t^{-1}$, so that by the theorem 7 of [2] we have $\|U_t\| \leqq 4K$ and $\|U_t^{-1}\| \leqq 4K$. By theorem 5 of [2], R_t and U_t commute with R_s and U_s for all $t, s > 0$. Applying WERMER's theorem 1 [5] to R_t and R_s, resp. to U_t and U_s, and using the fact that the product of two permutable positive selfadjoint operators is positive, and the product of two unitary operators is unitary, one obtains that $R_s R_t$ is a positive and $U_s U_t$ a circled scalar type operator. On the other hand

$$T_{t+s} = T_t T_s = R_t U_t R_s U_s = R_t R_s U_t U_s,$$

and $R_t R_s$, $U_t U_s$ commute with T_{t+s}. Thus, by virtue of lemma 2, we have $R_{t+s} = R_t R_s$, so that R_t is a semigroup of positive scalar type operators. To such a semi-group we can apply the considerations given in [3], p. 73, for the case of a semigroup of positive selfadjoint operators. To this aim, let $G_t(\sigma)$ be the spectral measure of R_t. If we put

$$R_1^{\frac{1}{2^n}} = \int_{\sigma(R_1)} \lambda^{\frac{1}{2^n}} G_1(d\lambda)$$

then by [2], lemma 6, $R_1^{\frac{1}{2^n}}$ is a scalar type operator whose spectral measure $G((-\infty, \lambda])$ is $G_1((-\infty, \lambda^{2^n}])$. On the other hand applying lemma 5 of [2] to $R_{\frac{1}{2^n}}$ we obtain that $G_1((-\infty, \mu]) = G_{\frac{1}{2^n}}((-\infty, \mu^{\frac{1}{2^n}}])$, so that $G((-\infty, \lambda]) = G_{\frac{1}{2^n}}((-\infty, \lambda])$; thus $R_1^{\frac{1}{2^n}}$ and $R_{\frac{1}{2^n}}$ have the same spectral measure, and so are identical. By the semigroup property, and by the functional calculus for scalar type operators, we obtain that

$$R_t = \int_{\sigma(R_1)} \lambda^t G_1(d\lambda) \tag{5}$$

for all numbers t in the set Q of the numbers of the form $\frac{m}{2^n}$ ($m = 1, 2, \ldots$; $n = 0, 1, \ldots$). From this formula and again from lemma 5 of [2] we obtain $G_t(\{0\}) = G_1(\{0\})$, hence in view of (3) and (4) we have $E_t(\{0\}) = E_1(\{0\})$ for all $t \in Q$. Consequently, we have for $s, t \in Q$

$$U_t U_s E_{t+s}(\{0\}) = U_t U_s E_1(\{0\}) = U_t U_s E_s(\{0\}) = U_t [U_s E_s(\{0\})] =$$
$$= U_t E_s(\{0\}) = U_t E_1(\{0\}) = U_t E_t(\{0\}) = E_t(\{0\}) = E_1(\{0\}) = E_{t+s}(\{0\})$$

and in view of the lemma,

$$U_{t+s} = U_t U_s \tag{6}$$

for all $s, t \in Q$. Let us put $U_0 = I$, and $U_{-s} = U_s^{-1}$ for $s \in Q$. In virtue of (6), $s \to U_s$ is an operator representation of the additive group $Q' = Q \cup \{-Q\} \cup \{0\}$ of all dyadically rational numbers. But, for all $s > 0$, U_s as a function of T_s

commutes with T_1, hence with R_1, and consequently we have

(7) $$U_s G_1(\sigma) = G_1(\sigma) U_s$$

for all Borel set $\sigma \subset \Omega$. Consider now the cartesian product Γ of the family of Borel sets σ in Ω, and of Q'. Define the „product"

$$(\sigma, s) \circ (\sigma', s') = ((\sigma \cap \sigma') \cup (\bar{\sigma} \cap \bar{\sigma}'), s + s'),$$

where the bar denotes complementation (i. e. $\bar{\sigma} = \Omega - \sigma$); Γ is then an abelian group with unit element $(\Omega, 0)$ and with the inverse $(\sigma, s)^{-1} = (\sigma, -s)$. For all $(\sigma, s) \in \Gamma$ put

$$W(\sigma, s) = [2G_1(\sigma) - I]\, U_s.$$

Using the fact that $G_1(\sigma)$ is a spectral measure, and the relations (6) and (7), one can easily verify that $(\sigma, s) \to W(\sigma, s)$ is an operator representation of our abelian group Γ. On the other hand the operators $W(\sigma, s)$ are uniformly bounded $[\|W(\sigma, s)\| \leqq \|2G_1(\sigma) - I\| 4K \leqq (8K+1)4K]$, so that we can apply Sz.-Nagy's theorem (in its form generalized to arbitrary abelian groups; see for instance [1], p. 222) and obtain a regular selfadjoint operator A, such that $A^{-1}W(\sigma, s)A$ are all unitary. For $s = 0$ we obtain that $A^{-1}[2G_1(\sigma) - I]A$ are unitary for all Borel sets $\sigma \subset \Omega$, thus $A^{-1}G_1(\sigma)A$ are orthogonal projections, and consequently, in virtue of (5), $A^{-1}R_tA$ is selfadjoint for all $t \in Q$. On the other hand putting $\sigma = \Omega$ and $t \in Q$ we get $W(\Omega, t) = U_t$, so that $A^{-1}U_tA$ are also unitary for all $t \in Q$. But for all $t > 0$ we have $R_tU_t = T_t = U_tR_t$, and hence

$$(A^{-1}R_tA)(A^{-1}U_tA) = A^{-1}T_tA = (A^{-1}U_tA)(A^{-1}R_tA).$$

Since, for all $t \in Q$, $A^{-1}R_tA$ is selfadjoint and $A^{-1}U_tA$ is unitary, their product $A^{-1}T_tA$ is a normal operator. But Q is dense on the positive semi-axis; using the strong continuity of T_t one obtains that $N_t = A^{-1}T_tA$ is normal for all $0 < t < \infty$, so that $\{T_t\}$ is similar to a semigroup of normal operators $\{N_t\}$, which finishes the proof of theorem 1.

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