On complete semi-modules.

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§ 1.

An algebraic structure is called a semi-module if it is an additive commutative regular semi-group. In this paper we assume that the considered semi-modules contain a zero-element.

RÉDEI [4] has defined the left-normal semi-modules similarly as the left-normal semi-groups. A sub-semi-module N of the semi-module S is called left-normal, if a compatible classification of S has the form

$$a_1+N, a_2+N, \ldots \qquad (a_1=0).$$

Since S is commutative, so every left-normal semi-module is at the same time right-normal too. Thus every left normal semi-module is a normal semi-module. Contrary to the case of modules, a sub-semi-module of a semi-module is not always normal. Consider for example the semi-module of the non-negative integers and a fixed integer $k(>0)$. The integers $n(\geqq k)$ form a semi-module, but this is not normal in the semi-module of the non-negative integers.

In a previous paper (WIEGANDT [6]) we have defined the complete structures. In this way a semi-module S is called complete, when it is a direct component of every semi-module which contains S as a normal semi-module.

In this paper we characterize the complete semi-modules, further we shall give all of the complete semi-modules, and we shall show that every semi-module is a sub-semi-module of a complete semi-module.

§ 2.

Theorem 1. *A semi-module S (with zero-element) is complete if, and only if, $nS=S$ for every natural number n.*

Remark. For comparison we mention that the complete abelian groups can be characterized by a similar condition, namely an abelian group A is complete (i. e. A is a direct component of every containing abelian group) if, and only if, $nA = A$ for every natural number n (see BAER [1]). By Theorem 1 the complete abelian groups are direct components of every containing semi-module too.

Proof. The condition $nS = S$ means that every equation

$$(1) \qquad nx = \sigma \qquad (\sigma \in S, n > 0 \text{ integer})$$

has a solution in S.

Assume that S is a complete semi-module. Let n be an arbitrary natural number and σ an arbitrary element of S. Consider the set of the pairs (k, μ) $(0 \leqq k \leqq n-1, \mu \in S)$, and define the addition in this set as follows:

$$(2) \qquad (k, \mu) + (l, \nu) = \left(\overline{k+l}, \left[\frac{k+l}{n}\right]\sigma + \mu + \nu\right)$$

where $\overline{k+l}$ means the least non-negative representant of the residue class ring mod n, $[a]$ means — as usual — "entier a". Denote this structure by $\bar{S}$. Clearly, the addition (2) is commutative, further according to the regularity of S the addition (2) is regular too. It is easy to see that for every integer k, l, m $(0 \leqq k, l, m \leqq n-1)$ we have

$$\left[\frac{k+\overline{l+m}}{n}\right] + \left[\frac{l+m}{n}\right] = \left[\frac{\overline{k+l}+m}{n}\right] + \left[\frac{k+l}{n}\right].$$

This implies that the function $k^l = \left[\frac{k+l}{n}\right]\sigma$ fulfills the condition (15') of RÉDEI [4]. Further we have $\left[\frac{k+0}{n}\right]\sigma = \left[\frac{0+k}{n}\right]\sigma = 0$ and so the function $k^l = \left[\frac{k+l}{n}\right]\sigma$ fulfills also the condition (12) of RÉDEI [4]. Thus by Corollary 2 of RÉDEI [4] $\bar{S}$ is an endomorphism-free Schreierian extension, namely by (2) an extension of S by the cyclic group of order n. So $\bar{S}$ is a semi-module, and the elements $(0, \tau)$ $(\tau \in S)$ form a normal semi-module in $\bar{S}$, which is isomorphic to S. Embed S into $\bar{S}$ and denote the element $(1, 0)$ by x. Since

$$(3) \qquad n(1, 0) = \left(\bar{n}, \left[\frac{n}{n}\right]\sigma\right) = (0, \sigma)$$

so x is a solution of the equation (1). Since S is normal in $\bar{S}$, therefore by the hypothesis S is a direct summand of $\bar{S}$,

$$\bar{S} = S + K.$$

Hence, the element x may be represented in one and only one way in the form

$$x = \eta + \varkappa \qquad (\eta \in S, \varkappa \in K)$$

and so

(4) $$nx = n\eta + n\varkappa.$$

On the other hand we have according to (3)

(5) $$nx = \sigma \in S.$$

Since $\bar{S}$ is a direct sum, so (4) and (5) implies $n\varkappa = 0$ and $nx = n\eta$. So $\eta (\in S)$ is a solution of the equation (1). Thus we have proved the necessity of the condition.

Conversely, assume that $nS = S$ is valid for every natural number n. Let T be an arbitrary semi-module which contains S as a normal semi-module. We show that S is a direct component of T.

Let S^* be the difference module of S in the sense of RÉDEI [5] Theorem 70. The elements of S^* are of the form $\sigma_1 - \sigma_2$ $(\sigma_1, \sigma_2 \in S)$. Let ξ_i $(i = 1, 2)$ be a solution of the equations

$$n\xi_i = \sigma_i \qquad (\sigma_i \in S, n > 0 \text{ integer})$$

in S. Since $\xi = \xi_1 - \xi_2$ $(\in S^*)$ is a solution of the equation

$$n\xi = \sigma_1 - \sigma_2$$

therefore we get $nS^* = S^*$ for S^*. Hence S^* is a complete abelian group. Embed S^* into the difference modul T^* of T. Since S^* is a complete abelian group, so S^* is a direct summand of T^*

$$T^* = S^* + D^*.$$

Thus the elements of $T (\subseteqq T^*)$ have the form $\varrho^* + \delta^*$ $(\varrho^* \in S^*, \delta^* \in D^*)$. Let ϱ^* be an arbitrary element which holds $\varrho^* = \varrho_1 - \varrho_2 \in T$ $(\varrho_1, \varrho_2 \in S)$ Obviously, in the classification of T according to S we have

$$\varrho_1 + S \subset \varrho_1 - \varrho_2 + S$$
$$\varrho_1 + S \subseteqq S$$

So the elements of S and of $\varrho_1 - \varrho_2 + S$ are in the same class and this class is exactly S. This implies $\varrho^* = \varrho_1 - \varrho_2 \in S$. Hence the elements of T can be represented in one and only one way in the form $\varrho + \delta^*$ $(\varrho \in S, \delta^* \in D^*)$, and this proves that S is a direct component of T.

To formulate our next theorem, we introduce the following terminology. By a group of type R we mean a group isomorphic to the additive group of all rational numbers, similarly by a semi-module of type N we mean a semi-module isomorphic to the additive group of all non-negative rationals. The complete semi-modules are completely described by

Theorem 2. *Every complete semi-module can be decomposed uniquely into the direct sum of groups of type R, semi-modules of type N and Prüfer's groups of type p^∞.*

Proof. Let S be an arbitrary complete semi-module. In S the elements which have an inverse, form an abelian group A.

Let a be an arbitrary element of A, and ξ_a any solution of the equation $n\xi = a$. Thus we have

$$n\xi_a - (n\xi_a) = 0,$$

and so

$$\xi_a + ((n-1)\xi_a - (n\xi_a)) = 0.$$

Hence ξ_a has an inverse, and by the construction of A we get $\xi_a \in A$. Thus A is an complete abelian group. Obviously A is normal in S. Hence A is a direct summand of S,

$$S = A + B.$$

Since A is a complete abelian group, so by a well-known theorem A is the direct sum of groups of type R and of type p^∞ (e. g. see KUROSH [3]). Clearly, B is a complete semi-module, and by the construction of A, the elements of B have not any inverse except the zero-element. This implies that the elements of B are of infinite order. We show that B can be decomposed uniquely into the direct sum of semi-modules of type N. The difference module B^* of B is a complete abelian group with elements of infinite order. Hence B^* can be decomposed uniquely into the direct sum of groups of type R. Thus B is also a direct sum, and each of its components is contained in a group of type R, further their components are obviously complete semi-modules by Theorem 1. We prove that their components are semi-modules of type N. It is sufficient to prove that, if a complete semi-module B_0 is contained in the module of the rational numbers, further if in B_0 only the zero-element has an inverse, then one of the following three cases occurs:

i) $B_0 = 0$,

ii) B_0 is the semi-module of the rationals $\geqq 0$,

iii) B_0 is the semi-module of the rationals $\leqq 0$.

Let $b\,(\neq 0)$ be an arbitrary element of B_0. We may suppose that b is an integer. At first we discuss the case $b > 0$. Since B_0 is a complete semi-module, so by Theorem 1 the equation $bx = b$ has a solution in B_0. Hence we get $x = 1 \in B_0$. Since $\frac{1}{n!}$ is the solution of the equation $n!\,x = 1$, so we have $\frac{1}{n!} \in B_0$. The numbers $0, 1, \frac{1}{2!}, \dots \frac{1}{n!}, \dots$ generate the semi-module of the non-negative rationals which is contained in B_0. B_0 does not contain negative rational numbers, because otherwise its inverse would be contained in B_0. Thus B_0 coincides with the semi-module of the non-negative rationals.

In the case $b<0$, we get on the same way the semi-module of the rationals $\leqq 0$, which is clearly a semi-module of type N.

Thus Theorem 2 is proved.

§ 3.

Concerning the embedding of a semi-module into a complete semi-module we have the following

Theorem 3. *Every semi-module can be embedded into a complete semi-module. In a complete semi-module which contains the semi-module S, there exists a smallest complete semi-module containing S, and this is uniquely determined up to isomorphism.*

Since the difference module of S can be embedded into a complete abelian group (see KULIKOV [2]), so the first statement is proved.

The proof of the further statements agree with the proof of the analogous theorem for abelian groups (see KUROSH [3]).

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