Note on complemented modular lattices of finite length.

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1. Introduction.

It is known, by a theorem of DEDEKIND ([1], p. 66, Theorem 2) and of BIRKHOFF ([1], p. 134, Theorem 2), respectively, that

(A) any non-modular lattice contains a sublattice isomorphic to the lattice of Fig. 1;

(B) any non-distributive modular lattice contains a sublattice isomorphic to the lattice of Fig. 2.

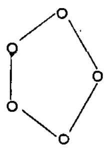

Fig. 1.

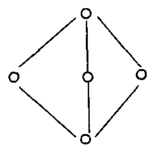

Fig. 2.

For complemented lattices of finite length the assertion (A) may be strengthened, by a theorem of DILWORTH ([2], p. 21), as follows:

(C) Any complemented non-modular lattice L of finite length contains a sublattice which includes the bounds[1]) of L and is isomorphic to the lattice of Fig. 1.

However, the assertion which may be analogously obtained from (B), is not true in general: there exist complemented non-distributive modular lattices L of finite length in which no sublattice isomorphic to the lattice of Fig. 2 includes the bounds of L. (Consider, for example, the lattice formed by the linear subspaces of the projective plane.) Accordingly, the problem arises to find necessary and sufficient conditions in order that such a lattice

[1]) By the *bounds* of a lattice we mean its least and greatest element (if existing). The least and the greatest element of a lattice (if existing) will be mostly denoted by o and i, respectively, but occasionally these letters will be supplied by subscripts.

L have a sublattice which is isomorphic to the lattice of Fig. 2 and at the same time includes the bounds of L. Our theorem in section 3 gives a solution of this problem; section 2 contains some preliminary lemmas.

2. Lemmas.

In this section we enumerate some results which either are known or may be easily obtained from known theorems.

In what follows, the height of an element x of a lattice of finite length will be denoted by $d(x)$. Then we have

Lemma 1. *Let L be a modular lattice of finite length. Then*

$$d(a \cap b) + d(a \cup b) = d(a) + d(b)$$

for each pair a, b of elements in L.

For the proof of this Lemma see [1], p. 67.

Lemma 2. *Let $\{p_1, \ldots, p_m\}$ be any set of points of a modular lattice of finite length. Then $d(p_1 \cup \cdots \cup p_m) \leqq m$.*

The assertion of this Lemma follows from Lemma 1 by induction.

Lemma 3. *Let L be a complemented modular lattice of length m. Then there exists a set $\{p_1, \ldots, p_m\}$ of points of L such that $\bigcup_{j=1}^{m} p_j = i$.*

Proof. By Theorem 6 on p. 105 of [1], the greatest element i of L may be represented as the join of points. Let P denote a set of points whose join is equal to i; from P we shall select, by induction, a subset of m elements with the required properties.

First we choose an arbitrary element p_1 of P. Then, if $p_1, \ldots, p_{j-1}$ are already selected, we single out an element p_j of P such that $(p_1 \cup \cdots \cup p_{j-1}) \cap p_j = o$. We continue this process until it is possible. Thus we get a subset $\bar{P} = \{p_1, p_2, \ldots, p_{j-1}, p_j, \ldots\}$ consisting of different points such that[2])

$$(p_1 \cup \cdots \cup p_{j-1}) \cap p_j \prec p_j \qquad (j = 2, 3, \ldots).$$

It follows, by the covering conditions, that

$$o \prec p_1 \prec \cdots \prec p_1 \cup \cdots \cup p_{j-1} \prec p_1 \cup \cdots \cup p_j \prec \cdots.$$

Since the length of L is m, this implies that if n denotes the number of elements of $\bar{P}$, then $n \leqq m$.

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[2]) By $x \prec y$ me mean that x is covered by y.

On the other hand, $n \geqq m$. Indeed, if $\bar{P}=\{p_1,\ldots,p_n\}$ and p is any element of P different from the elements p_j $(j=1,\ldots,n)$, then, by the construction of $\bar{P}$, $p \frown (p_1 \smile \cdots \smile p_n)=p$ which is equivalent to $p \smile (p_1 \smile \cdots \cdots \smile p_n)=p_1 \smile \cdots \smile p_n$. Hence

$$i=\bigcup_{p\in P} p=\bigcup_{j=1}^{n} p_j.$$

Consequently, by Lemma 2, $m=d(i)=d(p_1 \smile \cdots \smile p_n) \leqq n$, completing the proof.

A lattice is called *simple* if it has no non-trivial congruence relations. For simple modular lattices we have the following

Lemma 4. *Let L be a simple complemented modular lattice of finite length. Then to each pair p,q $(p \neq q)$ of its points there exists a third point r $(\neq p,q)$ which satisfies the equations $p \smile q = q \smile r = r \smile p$.*

Proof. It is known ([3], p. 89) that if a complemented modular lattice L of finite length is simple, then to each pair p,q $(p \neq q)$ of its points there exists a third point r in L such that $r < p \smile q$. It follows at once $p < p \smile r \leqq p \smile q$;[3]) moreover, $p \neq q$ implies $p \frown q = o \prec q$, whence we get, by the covering conditions, that $p \prec p \smile q$. Hence $p \smile r = p \smile q$. Similarly, $q \smile r = p \smile q$.

3. The theorem.

It is known ([1], pp. 120—121) that any complemented modular lattice L of finite length may be uniquely represented as a direct union

(1) $$L=P_0 \times P_1 \times \cdots \times P_r \qquad (r \text{ finite}),$$

where P_0 is a Boolean algebra and $P_1,\ldots,P_r$ are simple complemented modular, but non-distributive lattices. Using this result, we prove the following

Theorem. *Let L be any complemented modular lattice of finite length. Then L contains a sublattice which includes the bounds of L and is isomorphic to the lattice of Fig. 2 if and only if in its direct decomposition of the form* (1) *the component P_0 is the one-element lattice and the length of each P_j $(j=1,\ldots,r)$ is even.*

Proof. We prove our theorem in the following, equivalent form: *In a complemented modular lattice L of finite length the equation system*

(2) $$u \frown v = v \frown z = z \frown u = o,$$

(3) $$u \smile v = v \smile z = z \smile u = i$$

[3]) Since p, r both are points and $p \neq r$, $p = p \smile r$ is impossible.

is satisfied by some elements $u, v, z\,(\in L)$ *if and only if in the direct decomposition of the form* (1) *of* L *the component* P_0 *consists of a single element and each* P_j $(j=1,\dots,r)$ *is of even length.*

Now, according to (1), any elements u, v, z of L may be represented as

$$\left.\begin{aligned} u&=(u_0,\ u_1,\dots,\ u_r)\\ v&=(v_0,\ v_1,\dots,\ v_r)\\ z&=(z_0,\ z_1,\dots,\ z_r)\end{aligned}\right\}\qquad (u_j, v_j, z_j\in P_j;\quad j=0, 1,\dots, r).$$

Hence, if o_j and i_j denote the least and the greatest elements of P_j $(j=0, 1,\dots, r)$, respectively, then the equation system (2)—(3) is equivalent to the following equation system concerning the components of the elements u, v, z:

$$\left.\begin{aligned} &(4)\qquad u_j\cap v_j=v_j\cap z_j=z_j\cap u_j=o_j\\ &(5)\qquad u_j\cup v_j=v_j\cup z_j=z_j\cup u_j=i_j\end{aligned}\right\}\qquad (j=0, 1,\dots, r).$$

Proof of the necessity of the conditions. Let us assume that there exist elements u, v, z in L such that (2), (3) are satisfied. Then, as we have seen, the equations (4), (5) hold for the components of these elements.

Since P_0 is distributive, each element of P_0 has only one complement ([1], p. 134). This implies, with respect to (4) and (5), that at least two of the elements u_0, v_0, z_0 must be equal; by symmetry, we can assume $u_0=v_0$. It follows from (4) and (5) that

$$o_0=u_0\cap v_0=u_0\cap u_0=u_0=u_0\cup u_0=u_0\cup v_0=i_0.$$

Thus, P_0 consists actually of a single element.

Consider now the components P_j $(j=1,\dots, r)$. Each of these components is a modular lattice of finite length. Hence, by (4), (5) and by Lemma 1, we have

$$\left.\begin{aligned} d(u_j)+d(v_j)&=d(i_j)\\ d(v_j)+d(z_j)&=d(i_j)\\ d(z_j)+d(u_j)&=d(i_j)\end{aligned}\right\}\qquad (j=1,\dots, r).$$

By adding these equations we get

$$2(d(u_j)+d(v_j)+d(z_j))=3d(i_j)\qquad (j=1,\dots, r).$$

Thus we conclude that each $d(i_j)$ $(j=1,\dots, r)$ is divisible by 2, i. e. that the length of each P_j is even.

Proof of the sufficiency of the conditions. With regard to the equivalence of the two equation systems (2)—(3) and (4)—(5), respectively, it suffices to show that the following assertion is true: *In any simple complemented modular* *lattice P of even* ($\neq 0$) *length there exist elements a, b, c which satisfy the equations*

(6) $$a \cap b = b \cap c = c \cap a = o,$$

(7) $$a \cup b = b \cup c = c \cup a = i.$$

In order to prove this assertion, let us consider a simple complemented modular lattice P with $d(i) = 2n$, where n denotes a positive integer. Then, by Lemma 3, there exists a set $\{p_1, \ldots, p_{2n}\}$ of points of P such that

(8) $$\bigcup_{j=1}^{2n} p_j = i$$

and, by Lemma 2, $p_j \neq p_k$ for $j \neq k$ ($j, k = 1, \ldots, 2n$). It follows, by Lemma 4, that to each pair p_j, p_{n+j} ($j = 1, \ldots, n$) there exists a point q_j in P such that $q_j \neq p_j, p_{n+j}$ and

(9) $$p_j \cup p_{n+j} = p_j \cup q_j = p_{n+j} \cup q_j \qquad (j = 1, \ldots, n).$$

Now we define three elements a, b, c by

(10) $$a = \bigcup_{j=1}^{n} p_j, \qquad b = \bigcup_{j=1}^{n} p_{n+j}, \qquad c = \bigcup_{j=1}^{n} q_j$$

and we show that these elements satisfy the equations (6), (7).

Firstly, by (10) and (8), $a \cup b = i$. Next, by (10) and (9), we get

$$a \cup c = \bigcup_{j=1}^{n} (p_j \cup q_j) = \bigcup_{j=1}^{n} (p_j \cup p_{n+j}) = a \cup b = i$$

and, in the same way, $b \cup c = i$. Thus (7) is already verified. Furthermore, by Lemmas 1 and 2, we obtain

$$d(a \cap b) = d(a) + d(b) - d(a \cup b) = d(a) + d(b) - d(i) \leqq n + n - 2n = 0$$

which is equivalent to $a \cap b = o$. Similarly, $b \cap c = c \cap a = o$. Thus our assertion is proved.

References.

[1] G. Birkhoff, *Lattice theory*, Amer. Math. Soc. Coll. Publ., vol. 25, revised edition (New York, 1948).

[2] R. P. Dilworth, On complemented lattices, *Tôhoku Math. Journal*, 47 (1940), 18—23.

[3] H. Hermes, *Einführung in die Verbandstheorie*, Grundlehren der math. Wissenschaften in Einzeldarstellungen, Bd. 73 (Berlin—Göttingen—Heidelberg, 1955).

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