Bounds for the principal frequency of a membrane and the torsional rigidity of a beam

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1. We consider a simply connected or ring-shaped plane domain $D$ of area $A$, its boundary $C$ whose total length is $L$, its principal frequency $A$ and its torsional rigidity $P$.

The quantities $A$ and $P^{-1}$ may be defined as the minima of the expressions

\[
(1a) \quad \left( \frac{\iint (\text{grad } u)^2 \, d\sigma}{\iint u^2 \, d\sigma} \right)^\frac{1}{2}, \quad (1b) \quad \frac{\iint (\text{grad } u)^2 \, d\sigma}{4(\iint u \, d\sigma)^2},
\]

respectively, where $d\sigma$ is the surface element of $D$, the integrations are extended over $D$; the function $u$ is continuous in $D$, vanishes on $C$ and has piecewise continuous first derivatives in $D$.\(^1\)

We state that for a simply connected or ring-shaped domain

\[
(2a) \quad A \leq \sqrt{3} \frac{L}{A}, \quad (2b) \quad P^{-1} \leq \frac{L^2}{A^3} \quad \text{\(18\)}
\]

It is enough to show the validity of these inequalities for polygonal domains no two sides of which are parallel.\(^2\) The total statement follows hence by an argument of continuity.

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\(^1\) See e.g. G. Pólya—G. Szegő, *Isoperimetric Inequalities in Mathematical Physics* (Princeton, 1951), pp. 87 and 102—103.

\(^{18}\) (Note added on February 25, 1959.) The constants $\sqrt{3}$ and 1 on the right sides of (2a) and (2b), respectively, are not best possible. G. Pólya has shown that the precise upper bounds for $\Delta AL^{-1}$ and $P^{-1}A^3L^{-2}$ are $\frac{\pi}{2}$ and $\frac{3}{4}$, respectively; see his paper: Two more inequalities between physical and geometrical quantities (to be published in the *Journal of the Indian Math. Society*).

The inequalities (2) will be proved if one can find a particular function \( u \) for which the quantities (1a) and (1b) are less than (2a) resp. (2b). We shall see that such a function is the point function \( d(P) \) which is defined as the distance of the point \( P \) from the boundary \( C \). This function satisfies obviously the conditions imposed on the functions \( u \).

Let the vertices of \( C \) be \( A_1, A_2, \ldots, A_n \); the open line segment \( A_i A_{i+1} (A_{n+1} = A_1) \) will be denoted by \( a_i \). We may now define subdomains \( D_i \) and \( D'_i \) of \( D \) in the following way. The interior of \( D_i \) resp. \( D'_i \) contains those points of \( D \) the nearest point of the boundary to which lies on \( a_i \) resp. it is the point \( A_i \). The sum of the closures of the domains \( D_i \) and \( D'_i \) is \( D; D'_i \) is void if the inner angle at \( A_i \) is less than \( \pi \).

The level lines of \( d(P) \) are in \( D_i \) line segments parallel to \( a_i \), in \( D'_i \) circular arcs, whose centre is \( A_i \). In the interior of \( D_i \) or \( D'_i \) \(|\text{grad } d(P)| = 1\), and so

\[
\iint \left[\text{grad } d(P)\right]^2 d\sigma = A.
\]

Now the level line \( d(P) = \xi \) is identical with the boundary of an inner parallel point set of the domain \( D \). The length of this level line will be denoted by \( l(\xi) \). We may transform the double integral \( M_n = \iiint [d(P)]^n d\sigma \) into a simple one by dividing \( D \) into narrow stripes the boundaries of which are the level lines \( d(P) = \xi \) and the width of which is \( d\xi \):

\[
M_n = \iiint [d(P)]^n d\sigma = \int \frac{r^n}{\xi} l(\xi) d\xi
\]

where \( r \) is the radius of the greatest circle which can be inscribed in \( D \).

If \( n = 0 \) we have from (4) that

\[
\int_0^r l(\xi) d\xi = A.
\]

Let now the quantity \( b \) be defined by \( Lb = A \). As \( 0 \leq l(\xi) \leq L \) for \( 0 \leq \xi \leq r \),

\[
\int_0^r l(\xi) d\xi = A \leq \int_0^r L d\xi = Lr.
\]

So we have for \( n = 1, 2, \ldots \)

\[
\int_0^r \xi^n l(\xi) d\xi - \int_0^r \xi^n L d\xi = \int_0^r \xi^n l(\xi) d\xi - \int_0^r \{L - l(\xi)\} d\xi \geq 0
\]

\[
\geq b^n \int_0^r l(\xi) d\xi - b^n \int_0^r \{L - l(\xi)\} d\xi = b^n \{\int_0^r l(\xi) d\xi - \int_0^r L d\xi\} = 0
\]

\[3) \text{ A proof may be found in the paper by B. Sz.-NAGY, Über Parallelmengen nicht-}
by the definition of $b$. It follows that $M_n \geq \frac{b^{n+1}L}{n+1}$, hence

$$M_1 \geq \frac{A^2}{2L} \quad \text{and} \quad M_2 \geq \frac{A^3}{3L^3}$$

and from these

$$A \leq \left( \frac{\int \|\nabla d(P)\|^2 \, d\sigma}{\int \|d(P)\|^2 \, d\sigma} \right)^{1/2} \leq \left( \frac{A}{A^3/(3L^3)} \right)^{1/2} = \sqrt{3} \frac{L}{A}$$

resp.

$$P^{-1} \leq \frac{\int \|\nabla d(P)\|^2 \, d\sigma}{4 \left[ \int \|d(P)\| \, d\sigma \right]^2} \leq \frac{A}{4(A^2/2L)^2} = \frac{L^3}{A^3}.$$

2. There exists another upper estimate of $A$ and $P^{-1}$ for star-shaped domains, namely that of Pólya and Szegő. We consider the quantity $B_a = \int h^{-1} \, ds$ where $a$ is a point inside $D$ with respect to which $C$ is star-shaped, $h$ is the length of the perpendicular drawn from $a$ to the tangent at a variable point of $C$ where $ds$ is the line element. If $a$ varies and $B = \min B_a$, then $A \leq j\sqrt{B/2A}$ with $j = 2.40\ldots$, and $P^{-1} \leq BA^{-2}$.

It seems that for convex domains the estimate of Pólya and Szegő gives better results than (2). Yet e.g. for the pentagonal domain whose consecutive vertices are $(1,0)$, $(1,1)$, $(0,\varepsilon)$, $(-1,1)$, $(-1,0)$, $B$ tends to infinity as $\varepsilon \to 0$; on the other hand $L$ and $A$ remain bounded.

3. It may be noted that there does not exist a universal positive constant $c$ such that for any simply connected domain $A \geq cL/A$. (Contrary to the case when $D$ is convex.) For let us consider the domains

$$D_1(0 \leq x \leq 1, 0 \leq y \leq 1) \quad \text{and} \quad D_2(1 \leq x \leq 1 + \varepsilon^{-1}, 0 \leq y \leq \varepsilon).$$

In the case of the domain $D = D_1 + D_2$ we have $L/A = 2 + \varepsilon^{-1}$ and $A$ is bounded, for it is less than the principal frequency of the unit square.

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4) L. c. i) pp. 14—15 and 91—94.