Concerning the numerical integration of periodic functions of several variables

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Certain types of approximation formulae concerning the numerical integration of periodic functions of several variables of the form

$$(1) \qquad J=\left(\frac{1}{2\pi}\right)^m \int_0^{2\pi}\cdots\int_0^{2\pi} f(x_1,\ldots,x_m)\,dx_1\ldots dx_m$$

have been proposed by KOROBOV [1], HSU and LIN [2], respectively. Very recently S. H. MIN [3] has established an approximation formula whose degree of approximation is best possible as far as the order is concerned.

The object of this note is to show that a simple approximation formula with best possible degree of approximation can also be obtained by means of the method proposed in [2]. In the statement of MIN's result it is required that the integrand function should possess continuous partial derivatives of orders exceeding the number of variables contained. However, in our formula such an additional restriction becomes unnecessary.

As in [2], let $f(X)\equiv f(x_1,\ldots,x_m)$ be a function of period 2π in each of the variables x_j $(j=1,\ldots,m)$, and let $f(X)$ have continuous partial derivatives of the orders up to p $(p>3)$:

$$\frac{\partial^{\alpha_1+\cdots+\alpha_m}f}{\partial x_1^{\alpha_1}\ldots\partial x_m^{\alpha_m}}\equiv f_{\alpha_1\ldots\alpha_m}(X)\qquad(\alpha_j\geqq 0,\ \alpha_1+\cdots+\alpha_m\leqq p).$$

Define, as a modulus of continuity of order p,

$$\omega_p(\varrho)=\max|f_{\alpha_1\ldots\alpha_m}(x_1,\ldots,x_m)-f_{\alpha_1\ldots\alpha_m}(x_1',\ldots,x_m')|,$$

where the "maximum" is taken over all the compositions $(\alpha_1,\ldots,\alpha_m)$ subject to the condition $\alpha_1+\cdots+\alpha_m=p$ $(\alpha_j\geqq 0)$ and over all the points $(x_1,\ldots,x_m)$ and $(x_1',\ldots,x_m')$ with $\sum_j'(x_j-x_j')^2\leqq\varrho^2$.

By using the same device as in the proof of Theorem 1 in [2] we may establish the following

Lemma. *Let R denote a positive integer and let $\{\gamma_1, \dots, \gamma_m\}$ be a set of non-negative integers such that $\gamma_1 > \gamma_2 > \dots > \gamma_m$. Then for R large we have*

$$(2)\qquad \left| J - \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\,dt \right| = O\left(\left(\frac{1}{R}\right)^p \omega_p\left(\frac{1}{R}\right)\right),$$

where $\varphi(t) \equiv f(R^{\gamma_1} t, \dots, R^{\gamma_m} t)$, and the constant factor involved in the order estimation $O(\cdot)$ is independent of R.

Here we just sketch how to prove the lemma. Consider all the linear combinations of $T_j = R^{\gamma_j}$ with integer coefficients; we easily find that

$$\inf_{\nu < R^2} | n_1 T_1 + \dots + n_m T_m = R^{\gamma_m},$$

where the "infimum" is taken over all $(n_1, \dots, n_m)$ subject to the condition

$$1 \leqq n_1^2 + \dots + n_m^2 = \nu < R^2.$$

Then by exactly the same procedure as used in the derivation of (8) of [2] we may obtain

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\,dt = a_{0,\dots,0}(R) + O\left(\left(\frac{1}{R}\right)^p \omega_p\left(\frac{1}{R}\right)\right),$$

with $a_{0,\dots,0}(R) = C_{0,\dots,0} = J$. Hence we have the lemma.

We now proceed to construct an approximation formula. Take $\gamma_j = m - j$ in the lemma, so that

$$(3)\qquad \varphi(t) \equiv f(R^{m-1}t,\ R^{m-2}t, \dots, t).$$

Clearly $\varphi(t)$ is of period 2π in t. Suppose that $\varphi(t)$ has an absolutely convergent Fourier series

$$(4)\qquad \varphi(t) = \sum_{-\infty}^{\infty} C_n e^{int}.$$

Then we easily find that

$$(5)\qquad \frac{1}{N}\sum_{k=1}^{N} \varphi\left(\frac{2k\pi}{N}\right) = \frac{1}{N}\sum_{-\infty}^{\infty} C_n \sum_{k=1}^{N} \exp\left(\frac{2nk\pi i}{N}\right) = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)dt + \sum_{n\equiv 0 \,(\mathrm{mod}\, N)}{}' C_n,$$

where in Σ' is omitted the term C_0. Let us now estimate $|C_n|$ for $n = \pm N, \pm 2N, \dots$. Suppose that all the partial derivatives of $f(x_1, \dots, x_m)$ of order p are bounded in their absolute values. Then it follows that there is an absolute constant $A > 0$ such that

$$(6)\qquad \left(\frac{1}{R^{m-1}}\right)^p \left| \left(\frac{d}{dt}\right)^p \varphi(t) \right| < A \quad (0 \leqq t \leqq 2\pi)$$

Evidently, successive application of integration by parts to the integral

$$C_n = \frac{1}{2\pi}\int_0^{2\pi} \varphi(t) e^{-int}\, dt$$

will lead to the following estimate

$$(7) \qquad |C_n| \leqq \frac{1}{2\pi}\left|\frac{1}{-in}\right|^p \int_0^{2\pi} \left|\left(\frac{d}{dt}\right)^p \varphi(t)\cdot e^{-int}\right| dt \leqq \left|\frac{1}{n}\right|^p \cdot A\cdot R^{(m-1)p},$$

where we have utilized the periodicity of $\varphi^{(\nu)}(t)$ with $0 \leqq \nu \leqq p-1$. Thus we get

$$\left|\sum_{n\equiv 0 (\mathrm{mod}\, N)}' C_n\right| \leqq 2A\cdot\left(\frac{R^{m-1}}{N}\right)^p \sum_{k=1}^{\infty} \frac{1}{k^p} = 2A\cdot\left(\frac{R^{m-1}}{N}\right)^p \zeta(p).$$

Finally, taking $N = R^m$ and using (5), we find

$$\left|\frac{1}{N}\sum_{k=1}^{N} \varphi\left(\frac{2k\pi}{N}\right) - \frac{1}{2\pi}\int_0^{2\pi} \varphi(t)\, dt\right| \leqq 2A\cdot\zeta(p)\cdot\left(\frac{1}{R}\right)^p.$$

Comparing this with (2) and noticing that the absolute convergence of (4) is implied by (6) or (7), we are therefore led to the following

Theorem. *Let $N = R^m$ and let $\varphi(t)$ be defined by* (3) *and satisfy the boundedness condition* (6). *Then for N large we have*

$$(8) \qquad \left|\left(\frac{1}{2\pi}\right)^m \int_0^{2\pi}\cdots\int_0^{2\pi} f(X)\, dx_1 \ldots dx_m - \frac{1}{N}\sum_{k=1}^{N} f\left(\frac{2k\pi}{R}, \frac{2k\pi}{R^2}, \ldots, \frac{2k\pi}{R^m}\right)\right| \leqq$$
$$\leqq 2A\cdot\zeta(p)\left(\frac{1}{N}\right)^{p/m} + O\left[\left(\frac{1}{N}\right)^{p/m} \omega_p\left(\frac{1}{R}\right)\right],$$

where the constants in $O(\cdot)$ depend on the function f, but are independent of p and N.

Apparently we have here already arrived at a best possible result as far as the order $O(N^{-p/m})$ is concerned. As a matter of fact, it has been mentioned by GELFAND etc. [4] that one can apply the method of KOLMOGOROFF [5] to show that the order estimate $O(N^{-p/m})$ cannot be improved upon anyway. As regards the numerical calculation of a general multiple integral it is known that GNEDENKO [6] has proposed the problem (in certain connection with the Monte Carlo method) of improving the degree of approximation to the order $O(N^{-1})$. Now from our result we see that such a degree of approximation can actually be attained by assuming $p = m$; and on the other hand this will be not the case if it is assumed $p < m$.

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